Chapter 8: Statistics

8.1 Samples and Sampling Distributions

8.1 $\mu = 10 \quad \sigma_X^2 = 4 \quad n = 9$

(a) $$P[\bar{X}_9 < 9] = P\left[\frac{\bar{X}_9 - \mu}{\sigma_X/\sqrt{n}} < \frac{9-\mu}{\sigma_X/\sqrt{n}}\right] = P\left[\frac{\bar{X}_9 - 10}{2/\sqrt{9}} < \frac{9-10}{2/3}\right]$$

$$= 1 - Q\left(-\frac{3}{2}\right) = 0.0668$$

(b) $$P[\min(X_1, \ldots X_9) > 8] = P[X_1 > 8]\,P[X_2 > 8] \cdots P[X_9 > 8]$$

$$= P[X_1 > 8]^9$$

$$= Q\left(\frac{8-10}{2}\right)^9 = Q(-1)^9$$

$$= 0.2112$$

(c) $$P[\max(X_1, \ldots, X_9) < 12] = P[X_1 < 12] \cdots P[X_9 < 12]$$

$$= \left(1 - Q\left(\frac{12-10}{2}\right)\right)^9 = (1 - Q(1))^9$$

$$= 0.2112$$

(d) $$0.95 = P[|\bar{X}_n - 10| < 1] = P\left[\left|\frac{\bar{X}_n - 10}{2/\sqrt{n}}\right| < \frac{1}{2/\sqrt{n}}\right]$$

$$= P\left[-\frac{\sqrt{n}}{2} < \frac{\bar{X}_n - 10}{2/\sqrt{n}} < \frac{\sqrt{n}}{2}\right]$$

$$= P\left[-1.96 < \frac{\bar{X}_n - 10}{2\sqrt{n}} < 1.96\right]$$

$$\Rightarrow \sqrt{n} = 2(1.96) \qquad n = 4(1.96)^2 = 15.366 = 16$$

8.1 – continued –

Octave command to generate 100 samples of groups of 9

> normal_rnd (10, 4, 9, 100)

To find the sample mean of each group of 9:

> mean (normal_rnd (10, 4, 9, 100)

From sample of 100 we found:

$0.07 = \frac{7}{100}$ had values less than 9 vs. 0.0668 theory.

$0.19 = \frac{19}{100}$ had max of group < 12 vs. 0.2112

$0.18 = \frac{18}{100}$ had min of group > 8 vs. 0.2112

Max & MIN obtained using:

> max (normal_rnd (10, 4, 9, 100))

> min (normal_rnd (10, 4, 9, 100))

8.2 X exponential $\mu = 50$ $n = 25$ $\sigma^2 = \frac{1}{\lambda^2} = \mu^2 = 50^2$

a) $$P[|\bar{X}_{25} - 50| < 1] = P\left[\left|\frac{\bar{X}_{25}-50}{50/\sqrt{25}}\right| < \frac{1}{50/\sqrt{25}}\right]$$
$$= P\left[-\frac{1}{10} < \left|\frac{\bar{X}_{25}-50}{10}\right| < \frac{1}{10}\right]$$
$$= 0.07966$$

b) $$P[\max(X_1, \ldots, X_{25}) > 100] = 1 - P[\max(\ ) < 100]$$
$$= 1 - P[X_1 < 100]\,P[X_2 < 100] \ldots P[X_{25} < 100]$$
$$= 1 - (1 - e^{-100/50})^{25} = 1 - (1 - e^{-2})^{25}$$
$$= 0.9736$$

c) $$P[\min(X_1 \ldots X_{25}) < 25] = 1 - P[\min(X_1 \ldots X_{25}) > 25]$$
$$= 1 - P[X_1 > 25]^{25} = 1 - (e^{-25/50})^{25}$$
$$= 1 - e^{-25/2} = 1 - 3.73 \times 10^{-6}$$

d) $$0.90 = P[|\bar{X}_n - 50| < 5] = P\left[\left|\frac{\bar{X}_n - 50}{50/\sqrt{n}}\right| < \underbrace{\frac{5}{50/\sqrt{n}}}_{1.64}\right]$$
$$\frac{\sqrt{n}}{10} = 1.64$$
$$\sqrt{n} = 16.4 \qquad n = 269$$

e) Using approach in problem 8.1 (but generating exponential samples)

$0.08 = \frac{8}{100}$ samples were between 49 & 50 vs. 0.07966

$0.97 = \frac{97}{100}$ samples of max > 100 all samples of min < 25

8.3 X uniform $[-3,3]$ $n=50$ $\mu=0$ $\sigma^2=\frac{6^2}{12}=3$

a) $$P\left[|X_{50}|>0.5\right]=P\left[\left|\frac{X_{50}}{\sqrt{3}/\sqrt{50}}\right|>\underbrace{\frac{0.5}{\sqrt{3}/\sqrt{50}}}_{2.041}\right]=0.0206$$

b) $$P[\max(X_1,\ldots,X_{50})<2.5]=P[X_1<2.5]^{50}=\left(\frac{5.5}{6}\right)^{50}=0.0129$$

c) $$E[Y]=E[X^2]=\frac{1}{6}\int_{-3}^{3}x^2\,dx=3$$
$$E[Y^2]=E[X^4]=\frac{1}{6}\int_{-3}^{3}x^4\,dx=\frac{81}{5}$$
$$VAR[Y]=E[Y^2]-E[Y]^2=\frac{81}{5}-9=\frac{36}{5}$$
$$P[\bar{Y}_{50}>3]=P\left[\frac{\bar{Y}_{50}-3}{6/\sqrt{5}\cdot\sqrt{50}}>\frac{3-3}{6/\sqrt{5}\sqrt{50}}\right]=Q(0)$$
$$=1/2$$

d) In 100 samples of sample means in groups of 50

$0.0\_=\frac{\_}{100}$ were >0.5 vs. 0.0206

none of the max's were <2.5 vs. 0.0129

$0.50=\frac{50}{100}$ of the sample means of $Y^2<3$ vs 0.50.

8.4 $X \quad \mu = 2 \quad \sigma^2 = 2 \quad n = 16$

a) $P[\bar{X}_{16} > 2.5] = P\left[\frac{\bar{X}_{16} - 2}{\sqrt{2}/4} > \frac{2.5-2}{\sqrt{2}/4}\right] =$

$= Q(\sqrt{2}) = 0.0786$

b) $P[|\bar{X}_{16} - 2| > 0.5] = P\left[\frac{|\bar{X}_{16} - 2|}{\sqrt{2}/4} > \frac{0.5}{\sqrt{2}/4}\right]$

$= 2Q(\sqrt{2}) = 0.1572$

c) $0.95 = P[|\bar{X}_n - 2| > 0.5] = P\left[\frac{|\bar{X}_n - 2|}{\sqrt{2}/\sqrt{n}} > \underbrace{\frac{0.5}{\sqrt{2}/\sqrt{n}}}_{1.96}\right]$

$\frac{0.5\sqrt{n}}{\sqrt{2}} = 1.96$

$n = ((1.96)(2)\sqrt{2})^2$
$= 30.73$
$= 31$

d) Use method in Problem 8.1

8.5 X exponential $\frac{1}{\lambda}=\frac{1}{4}$ $n=9$ $\sigma^2=\frac{1}{\lambda^2}=\frac{1}{16}$

(a) $P[\,|\hat{\lambda}_1-4|>1] = P[\,|\frac{1}{\bar{X}_9}-4|>1] = 1-P[\,|\frac{1}{\bar{X}_9}-4|<1]$

$$= 1-P[-1<\frac{1}{\bar{X}_9}-4<1]$$

$$= 1-P[3<\frac{1}{\bar{X}_9}<5] = 1-P[3\bar{X}_9<1<5\bar{X}_9]$$

$$= 1-P[\tfrac{1}{5}<\bar{X}_9<\tfrac{1}{3}]$$

$$= 1-P\left[\frac{\frac{1}{5}-\frac{1}{4}}{1/\sqrt{16}\sqrt{3}}<\bar{X}_9<\frac{\frac{1}{3}-\frac{1}{4}}{1/\sqrt{16}\sqrt{3}}\right]$$

$$= 1-P\left[-\frac{\sqrt{3}}{5}<\bar{X}_9<\frac{1}{\sqrt{3}}\right]$$

$$\approx 1-0.3536 = 0.6463$$

(b) $$\hat{\lambda}_2 = \frac{1}{9\min(X_1,\dots,X_9)}$$

$$P[\,|\hat{\lambda}_2-4|>1] = P[\hat{\lambda}_2<3 \text{ or } \hat{\lambda}_2>5]$$

$$= P\left[\frac{1}{9\min()}<3\right] + P\left[\frac{1}{9\min()}>5\right]$$

$$= P\left[\tfrac{1}{27}<\min(\,)\right] + P\left[\tfrac{1}{45}>\min(\,)\right]$$

$$= P[X>\tfrac{1}{27}]^9 + 1-P[X>\tfrac{1}{45}]^9$$

$$= (e^{-4/27})^9 + 1-(e^{-4/45})^9$$

$$= e^{-4/3}+1-e^{-4/5} = 0.814$$

(c) Out of 100 samples of minimum of group of 9

$\frac{80}{100}$ had values <3 or >5.

8.6 (a) X uniform in $[0, \theta]$ $\quad E[X] = \frac{\theta}{2}$

$$\hat{m}_1 = \frac{1}{n}\sum_{j=1}^{n} X_j = \frac{\theta}{2}$$

$$\Rightarrow \hat{\theta} = 2\hat{m}_1$$

(b)

$$E[\hat{\theta}] = E\left[2\frac{1}{n}\sum_{j=1}^{n} X_j\right] = 2\frac{1}{n}\sum_{j=1}^{n} E[X] = 2E[X].$$

$$VAR[\hat{\theta}] = E\left[(\hat{\theta} - 2E[X])^2\right] = E\left[\left(\frac{2}{n}\sum_{j=1}^{n} X_j - 2E[X]\right)^2\right]$$

$$= \frac{4}{n^2} E\left[\left(\sum_{j=1}^{n}(X_j - E[X])\right)^2\right]$$

$$= \frac{4}{n^2} E\left[\sum_{j=1}^{n}\sum_{i=1}^{n}(X_j - E[X])(X_i - E[X])\right]$$

$$= \frac{4}{n^2} \sum_{j=1}^{n} E[(X_j - E[X])^2] + \sum_{\substack{i \\ i \neq j}}\sum_{j} \underbrace{E[X_j - E[X]]}_{0}\, E[X_i - E[X_i]]$$

$$= \frac{4}{n^2} n VAR[X]$$

$$= \frac{4}{n} VAR[X].$$

8.7 X Gamma α and $\beta = 1/\lambda$.

a) $$\hat{m}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i \approx \frac{\alpha}{\lambda} \Rightarrow \alpha = \lambda \hat{m}_1$$

$$\hat{m}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \approx VAR[X] + E[X]^2 = \frac{\alpha}{\lambda^2} + \frac{\alpha^2}{\lambda^2}$$

$$\Rightarrow \hat{m}_2 = \frac{\lambda \hat{m}_1}{\lambda^2} + \frac{\lambda^2 \hat{m}_1^2}{\lambda^2} = \frac{\hat{m}_1}{\lambda} + \hat{m}_1^2$$

$$\hat{m}_2 - \hat{m}_1^2 = \frac{\hat{m}_1}{\lambda} \qquad \lambda = \frac{\hat{m}_1}{\hat{m}_2 - \hat{m}_1^2}$$

$$\alpha = \lambda \hat{m}_1 = \frac{\hat{m}_1^2}{\hat{m}_2 - \hat{m}_1^2}$$

b) As n becomes large $\hat{m}_1$ and $\hat{m}_2$ approach $E[X]$ and $E[X^2]$ and so the estimators approach their true values.

Note: Means are known.

8.8

(a)

$$E[\hat{C}_{XY}] = E\left[\frac{1}{n}\sum_{j=1}^{n}(X_j-\mu_1)(Y_j-\mu_2)\right]$$

$$= \frac{1}{n}\sum_{j=1}^{n}\underbrace{E[(X_j-\mu_1)(Y_j-\mu_2)]}_{COV(X,Y)}$$

$$= \frac{1}{n}\, n\, COV(X,Y) = COV(X,Y).$$

$$E[(\hat{C}_{XY} - COV(X,Y))^2] = E\left[\left(\frac{1}{n}\sum(X_j-\mu_1)(Y_j-\mu_2) - COV(X,Y)\right)^2\right]$$

$$= E\left[\frac{1}{n^2}\sum_j\sum_i\{(X_j-\mu_1)(Y_j-\mu_2) - COV(X,Y)\}\{(X_i-\mu_1)(Y_i-\mu_2) - COV(X,Y)\}\right]$$

$i \neq j$ cross product terms have zero expected value

$$= \frac{1}{n^2}\sum_j E\left[\left((X_j-\mu_1)(Y_j-\mu_2) - COV(X,Y)\right)^2\right]$$

$$= \frac{1}{n^2}\sum_j \Big\{E[(X_j-\mu_1)^2(Y_j-\mu_2)^2] - 2E[(X_j-\mu_1)(Y_j-\mu_2)]COV(X,Y) + COV^2(X,Y)\Big\}$$

$$= \frac{1}{n}\left[E[(X-\mu_1)^2(Y-\mu_2)^2] - COV^2(X,Y)\right]$$

(b) If $E[(X-\mu_1)^2(Y-\mu_2)^2]$ is bounded then variance of estimator approaches zero as $n \to \infty$.

8.9 Means unknown.

a)
$$\hat{K}_{XY} = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \bar{X}_n)(Y_j - \bar{Y}_n)$$

$$= \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \mu_1 + \mu_1 - \bar{X}_n)(Y_j - \mu_2 + \mu_2 - \bar{Y}_n)$$

$$= \frac{1}{n-1}\sum_{j}\Big[(X_j-\mu_1)(Y_j-\mu_2) + (\mu_1 - \bar{X}_n)(Y_j - \mu_2) + (X_j-\mu_1)(\mu_2-\bar{Y}_n) + (\mu_1-\bar{X}_n)(\mu_2-\bar{Y}_n)\Big]$$

$$= \frac{1}{n-1}\Big\{\sum_j (X_j-\mu_1)(Y_j-\mu_2) + (\mu_1-\bar{X}_n)\sum_j (Y_j-\mu_2) + (\mu_2-\bar{Y}_n)\sum_j (X_j-\mu_1) + (\mu_1-\bar{X}_n)(\mu_2-\bar{Y}_n)\Big]$$

$$= \frac{1}{n-1}\sum_j\Big\{(X_j-\mu_1)(Y_j-\mu_2) - (\mu_1-\bar{X}_n)(\mu_2-\bar{Y}_n)\Big]$$

$$E[\hat{K}_{XY}] = \frac{1}{n-1}\sum_j\Big\{COV(X,Y) - \frac{1}{n^2}\underbrace{\sum_i (X_i-\mu_1)\sum_j (Y_j-\mu_2)}_{n\,COV(X,Y)}\Big\}$$

$$= \frac{n}{n-1}\Big\{1-\frac{1}{n}\Big\}COV(X,Y)$$

$$= COV(X,Y)$$

b) as n becomes large $\bar{X}_n \to E[X]$ and $\bar{Y}_n \to E[Y]$

so the estimator approaches the estimator in problem 8.8.

8.10 $W = \min(X_1 \ldots X_n)$ $\quad Z = \max(X_1 \ldots X_n)$

a) $P[\min(X_1 \ldots X_n) > x] = P[X_1 > x, X_2 > x, \ldots, X_n > x]$

$$= P[X > x]^n$$

$$\Rightarrow F_W(x) = 1 - P[X > x]^n$$

$$f_W(x) = n F_X(x)^{n-1} f_X(x)$$

b) $P[\max(X_1 \ldots X_n) \le x] = P[X_1 \le x, \ldots, X_n \le x]$

$$= P[X < x]^n = F_X^n(x)$$

$$\Rightarrow F_Z(x) = F_X^n(x)$$

$$f_Z(x) = n F_X^{n-1}(x) f_X(x).$$

8.2 Parameter Estimation

8.11

$$E[(\hat{\theta}-\theta)^2] = E[(\hat{\theta} - E[\hat{\theta}] + E[\hat{\theta}] - \theta)^2]$$

$$= E\left[(\hat{\theta} - E[\hat{\theta}])^2 + 2(\hat{\theta} - E[\hat{\theta}])(E[\hat{\theta}] - \theta) + (E[\hat{\theta}] - \theta)^2 \right]$$

$$= VAR[\hat{\theta}] + 2\underbrace{E[\hat{\theta} - E[\hat{\theta}]]}_{0}(E[\hat{\theta}] - \theta) + \underbrace{(E[\hat{\theta}] - \theta)}_{B(\hat{\theta})}{}^2$$

$$= VAR[\hat{\theta}] + B(\hat{\theta})^2$$

8.12 X_i Poisson, $\alpha = 4$

a) $E[\hat{\alpha}_1] = E\left[\frac{X_1 + X_2}{2}\right] = \frac{1}{2}E[X_1] + \frac{1}{2}E[X_2] = \alpha$ unbiased

$$VAR[\hat{\alpha}_1] = VAR\left[\frac{X_1 + X_2}{2}\right] = \frac{VAR[X]}{2} = \frac{\alpha}{2}$$

b) $E[\hat{\alpha}_2] = E\left[\frac{X_3 + X_4}{2}\right] = \alpha$ unbiased

$$VAR[\hat{\alpha}_2] = VAR\left[\frac{X_3 + X_4}{2}\right] = \frac{VAR[X]}{2} = \frac{\alpha}{2}$$

c) $E[\hat{\alpha}_3] = E\left[\frac{X_1 + 2X_2}{3}\right] = \frac{1}{3}E[X_1] + \frac{2}{3}E[X_2] = \alpha$ unbiased

$$VAR[\hat{\alpha}_3] = E\left[\left(\frac{1}{3}X_1 + \frac{2}{3}X_2 - \alpha\right)^2\right] = E\left[\left(\frac{1}{3}X_1 - \frac{1}{3}\alpha + \frac{2}{3}X_2 - \frac{2}{3}\alpha\right)^2\right]$$

$$= \frac{1}{9}E[(X_1 - \alpha)^2] + \frac{4}{9}E[(X_2 - \alpha)^2] + 0$$

$$= \frac{1}{9}\alpha + \frac{4}{9}\alpha = \frac{5}{9}\alpha$$

d) $E[\hat{\alpha}_4] = E\left[\frac{X_1 + X_2 + X_3 + X_4}{4}\right] = E[X] = \alpha$ unbiased

$$VAR[\hat{\alpha}_4] = \frac{VAR[X]}{4} = \frac{\alpha}{4}$$

8.13

(a) $E[\hat{\Theta}] = E[p\hat{\Theta}_1 + (1-p)\hat{\Theta}_2] = pE[\hat{\Theta}_1] + (1-p)E[\hat{\Theta}_2] = p\theta + (1-p)\theta = \theta$

$\Rightarrow$ unbiased.

(b) $$\frac{d}{dp}\left[E\left[\left(p\hat{\Theta}_1 + (1-p)\hat{\Theta}_2 - \theta\right)^2\right]\right] = E\left[2\left(p\hat{\Theta}_1 + (1-p)\hat{\Theta}_2\right)\left(\hat{\Theta}_1 - \hat{\Theta}_2\right)\right] = 0$$

$$0 = pE[\hat{\Theta}_1(\hat{\Theta}_1 - \hat{\Theta}_2)] + (1-p)E[\hat{\Theta}_2(\hat{\Theta}_1 - \hat{\Theta}_2)]$$

$$= pE[\hat{\Theta}_1(\hat{\Theta}_1 - \hat{\Theta}_2)] + E[\hat{\Theta}_2(\hat{\Theta}_1 - \hat{\Theta}_2)] - pE[\hat{\Theta}_2(\hat{\Theta}_1 - \hat{\Theta}_2)]$$

$$\Rightarrow \quad p = \frac{E[\hat{\Theta}_2(\hat{\Theta}_1 - \hat{\Theta}_2)]}{E[\hat{\Theta}_2(\hat{\Theta}_1 - \hat{\Theta}_2)] - E[\hat{\Theta}_1(\hat{\Theta}_1 - \hat{\Theta}_2)]}$$

(c) $\hat{\Theta}_1 = \dfrac{X_1 + X_2}{2} \quad \hat{\Theta}_2 = \dfrac{X_3 + X_4}{2}$ Note that $E[\hat{\Theta}_1\hat{\Theta}_2] = E[\hat{\Theta}_1]E[\hat{\Theta}_2]$ since X_1, X_2 are indep of X_3, X_4

$$p = \frac{E[\hat{\Theta}_2]E[\hat{\Theta}_1] - E[\hat{\Theta}_2^2]}{E[\hat{\Theta}_2]E[\hat{\Theta}_1] - E[\hat{\Theta}_2^2] - E[\hat{\Theta}_1^2] + E[\hat{\Theta}_1]E[\hat{\Theta}_2]}$$

$$= \frac{\alpha^2 - (\alpha + \alpha^2)}{\alpha^2 - (\alpha + \alpha^2) - (\alpha + \alpha^2) + \alpha^2}$$

$$= \frac{-\alpha}{-2\alpha} = \frac{1}{2}$$

(d) $$E[\hat{\Theta}_1\hat{\Theta}_4] = E\left[\tfrac{1}{2}(X_1 + X_2) \cdot \tfrac{1}{4}(X_1 + X_2 + X_3 + X_4)\right]$$

$$= \frac{1}{8}\underbrace{E[(X_1 + X_2)^2]}_{2\alpha + 4\alpha^2} + \frac{1}{8}\underbrace{E[(X_1 + X_2)]}_{2\alpha}\underbrace{E[(X_3 + X_4)]}_{2\alpha}$$

$$= \frac{1}{4}\alpha + \alpha^2$$

8.13d — continued — from part (b)

$$\rho = \frac{E[\hat{\theta}_4 \hat{\theta}_1] - E[\hat{\theta}_4^2]}{E[\hat{\theta}_4 \hat{\theta}_1] - E[\hat{\theta}_4^2] - E[\hat{\theta}_1^2] + E[\hat{\theta}_4 \hat{\theta}_1]}$$

$$E[\hat{\theta}_4^2] = VAR[\hat{\theta}_4] + E[\hat{\theta}_4]^2 = \frac{\alpha}{4} + \alpha^2$$

$\therefore$ $\rho = 0$ $\Rightarrow$ Estimator 1 is not used in combined estimator because its terms already accounted for.

(e)

$$E[\hat{\theta}_1] = E[X] \qquad E[\hat{\theta}_2] = E[X^2]$$

$$VAR[X] = E[X^2] - E[X]^2$$

An obvious choice to estimate $VAR[X]$ is

$$\hat{\theta}_3 = \hat{\theta}_2 - \hat{\theta}_1^2$$

then

$$E[\hat{\theta}_3] = E[\hat{\theta}_2] - E[\hat{\theta}_1^2] = E[X^2] - \left(VAR[\hat{\theta}_1] + E[\hat{\theta}_1]^2\right)$$

$$= E[X^2] - E[X]^2 \underbrace{- VAR[\hat{\theta}_1]}_{\text{bias}}$$

8.14 $Y = \theta + N$ $\quad$ N unif in $[0,2]$ $\quad$ $Y_i = \theta + N_i$ $\quad$ $i = 1, \ldots, n$ $\quad$ N_i iid.

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

$$E[\bar{Y}_n] = E\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] = \frac{1}{n} E\left[\sum_{i=1}^{n} (\theta + N_i)\right] = \theta + E[N_i]$$

$$= \theta + 1 \Rightarrow \text{biased estimator}$$

$$E[(\bar{Y}_n - \theta)^2] = VAR[\bar{Y}_n] + B(\bar{Y}_n)^2 = \frac{1}{n} VAR[Y] + 1$$

$$= \frac{1}{n}\frac{2^2}{12} + 1 = \frac{1}{3n} + 1.$$

8.15 X Poisson $\alpha = 2$ req/min

ⓐ $\hat{p}_0 = e^{-\alpha}$ $\quad$ $\hat{\alpha} = \ln 1/\hat{p}_0$ $\quad$ # of zero-arrival intervals.

Binomial with parameter n + $p_0 = e^{-\alpha}$

ⓑ $$E[\hat{\alpha}] = -E[\ln \hat{p}_0] = -E\left[\ln \frac{k_0}{n}\right]$$

$$= -\sum_{j=0}^{n} \binom{n}{j} p_0^{\,j} (1-p_0)^{n-j} \ln \frac{j}{n}$$

$$= -\sum_{j=0}^{n} \binom{n}{j} (e^{-2})^j (1-e^{-2})^{n-j} \ln \frac{j}{n}$$

$$= \underbrace{-\ln \frac{0}{n}}_{\infty} (1-e^{-2})^n + \text{other terms}$$

if no zero-arrivals intervals occur estimator decides arrival rate is infinite.

ⓒ $$MSE[\hat{\alpha}] = VAR[\hat{\alpha}] + B(\hat{\alpha})^2 = \text{infinity}.$$

ⓓ As $n \to \infty$ $\quad$ $\hat{p}_0 \to p_0 \Rightarrow \hat{\alpha} = \ln \frac{1}{\hat{p}_0} \to \alpha$

$\Rightarrow \hat{\alpha}$ is consistent.

Sample Mean Estimator

8.16

```
> x = poisson_rnd(2, 20, 100);      generates 100 groups of 10 poisson samples
> mean(mean(x));                    sample mean of group means
ans = 2.0585
> std(mean(x))^2;                   Sample variance of group means
ans = 0.098488
```

∴ sample mean estimator works well.

Zero-Count Estimator

Let x be poisson samples from above

```
> y = min(x, 1);       indicator function for non-zero arrival
> sum(y);              # of nonzero arrivals
> 20.- sum(y);         # of zero counts in 20 minutes
```

Histogram for 100 samples are

k_0	# occurences	estimate value
0	10	∞
1	22	$-\ln 1/20 = +3$
2	24	$-\ln 2/20 = 2.3$
3	15	$-\ln 3/20 = 1.90$
4	19	$-\ln 4/20 = 1.61$
5	6	$-\ln 5/20 = 1.39$
6	3	$-\ln 6/20 = 1.20$
7	1	$-\ln 7/20 = 1.05$

sample mean of 90 nonzero estimates 2.1465

Excluding the intervals where no arrivals occur leads to a viable estimator.

8.17 $\hat{p} = \frac{k}{n}$ $\quad \hat{\sigma}_n^2 = \hat{p}(1-\hat{p}) = \frac{k}{n}\left(1-\frac{k}{n}\right)$

(a)
$$E[\sigma_n^2] = E\left[\frac{k}{n}\left(1-\frac{k}{n}\right)\right] = E\left[\frac{k}{n} - \frac{k^2}{n^2}\right] = \frac{1}{n}E[k] - \frac{1}{n^2}E[k^2]$$
$$= \frac{1}{n}np - \frac{1}{n^2}\left[npq + (np)^2\right]$$
$$= p - \frac{pq}{n} - p^2 = \underbrace{(p-p^2)}_{\substack{p(1-p) \\ \text{variance of} \\ \text{Bernoulli}}} - \underbrace{\frac{pq}{n}}_{\text{bias.}}$$

(b) as $n \to \infty$ $\hat{p} = \frac{k}{n} \to p$

$\therefore$ $\hat{\sigma}_n^2 \to p(1-p)$ as well

$\therefore$ $\hat{\sigma}_n^2$ is consistent.

(c)
$$E[\sigma_n^2] = p(1-p) - \frac{p(1-p)}{n} = p(1-p)\underbrace{\left(1-\frac{1}{n}\right)}_{\frac{n-1}{n}}$$

$\Rightarrow c = \frac{n}{n-1}$

(d)
$$MSE[\hat{\sigma}_n^2] = E\left[\left(\hat{\sigma}_n^2 - p(1-p)\right)^2\right]$$
$$= E\left[\left(\hat{\sigma}_n^2\right)^2\right] - 2\underbrace{E[\hat{\sigma}_n^2]}_{\frac{p(1-p)(n-1)}{n}}p(1-p) + p^2(1-p)^2$$

$$E\left[\left(\hat{\sigma}_n^2\right)^2\right] = E\left[\left(\frac{k}{n}\right)^2\left(1-\frac{k}{n}\right)^2\right] = E\left[\frac{k^2}{n^2}\left(1-\frac{2k}{n}+\frac{k^2}{n^2}\right)\right]$$
$$= \frac{1}{n^2}E[k^2] - \frac{2}{n^3}E[k^3] + \underbrace{\frac{1}{n^4}E[k^4]}_{\text{Need first 4 moments!}}$$

– These moments can be found from the generating function discussed in chapter 4.

8.18 $\hat{\Theta} = \max(X_1, \ldots, X_n)$ $\quad X_i$ uniform in $[0, \theta]$

a) $f_{\hat{\theta}}(x) = n F_X(x)^n f_X(x) = n \left(\frac{x}{\theta}\right)^n \frac{1}{\theta} \quad 0 < x < \theta$

b) $E[\hat{\theta}] = \int_0^\theta x f_{\hat{\theta}}(x)\,dx = \frac{n}{\theta^{n+1}} \int_0^\theta x^{n+1}\,dx = \frac{n}{\theta^{n+1}} \frac{\theta^{n+2}}{n+2} = \frac{n}{n+2}\theta$

$B[\hat{\theta}] = E[\hat{\theta}] - \theta = -\frac{2}{n+2}\theta$

c) $E[\hat{\theta}^2] = \frac{n}{\theta^{n+1}} \int_0^\theta x^{n+2}\,dx = \frac{n}{\theta^{n+1}} \frac{\theta^{n+3}}{n+3} = \frac{n}{n+3}\theta^2$

$VAR[\hat{\theta}] = \frac{n}{n+3}\theta^2 - \left(\frac{n}{n+2}\theta\right)^2 = \theta^2\left[\frac{n}{n+3} - \frac{n^2}{(n+2)^2}\right]$

$MSE[\hat{\theta}] = VAR[\hat{\theta}] + \frac{4\theta^2}{(n+2)^2} \to 0$ as $n \to \infty$ so estimator is consistent.

d) $\hat{\theta}' = \frac{n+2}{n} \max(X_1, \ldots, X_n)$ is unbiased.

e) $\hat{\theta}$ in 100 trials of 20 observations had a sample mean of 4.7923 vs theoretical of 4.5455

$\hat{\theta}'$ in same 100 trials had sample mean of 5.2716

8.19 $1-F_X(x) = \frac{\theta^k}{x^k} \quad x \geq \theta$ Pareto.

$\therefore \hat{\theta} = \min(X_1, \ldots, X_n) \qquad f_{\hat{\theta}}(x) = n[1-F_X(x)]^{n-1} f_X(x).$

$$= n\left(\frac{\theta^k}{x^k}\right)^{n-1} k \frac{\theta^k}{x^{k+1}}$$

ⓐ $E[\hat{\theta}] = nk\int_\theta^\infty x \frac{\theta^{kn}}{x^{kn}} \frac{1}{x}\, dx = nk\int_\theta^\infty \frac{\theta^{kn}}{x^{kn}} dx$

$$= nk\theta^{kn} \left.\frac{x^{-kn+1}}{-kn+1}\right|_\theta^\infty = -\frac{nk\theta^{kn}}{(1-kn)\theta^{kn-1}} = \left(\frac{nk}{nk-1}\right)\theta$$

$$= \left(1+\frac{1}{nk-1}\right)\theta = \theta + \underbrace{\frac{1}{nk-1}\theta}_{\text{bias.}}$$

ⓑ $E[\hat{\theta}^2] = nk\int_\theta^\infty x^2 \frac{\theta^{kn}}{x^{kn}} \frac{1}{x}\, dx = nk\theta^{kn} \left.\frac{x^{-kn+2}}{-kn+2}\right|_\theta^\infty = \frac{nk\theta^2}{nk-2}$

$$MSE[\hat{\theta}] = VAR[\hat{\theta}] + B(\hat{\theta})^2$$

$$= \underbrace{\frac{nk\theta^2}{nk-2}}_{E[\hat{\theta}^2]} + \underbrace{\left(\frac{\theta}{nk-1}\right)^2}_{B(\hat{\theta})^2} \underbrace{- \left(\frac{nk}{nk-1}\right)^2\theta^2}_{-E[\hat{\theta}]^2}$$

$$= \frac{nk\theta^2}{nk-2} + \frac{\theta^2 - (nk)^2\theta^2}{(nk-1)^2} \rightarrow \theta^2 - \theta^2 = 0.$$

ⓒ $\therefore \hat{\theta}$ is consistent.

8.20

Sample Mean Over 100 Samples

Group Size	Unbiased	Biased
5	1,0163	0,84690
10	0,98573	0,86157
20	0,97089	0,92235

8.21 We are interested in the variance of the sample variance estimator

$$\hat{\sigma}_N^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$

Note that the sample variance of X_i and $X_i - E[X]$ will have the same distribution. For this reason we can assume that X_i is zero mean, $m_X = 0$.

Consider:

$$s = \sum_{i=1}^{n}(X_i - \bar{X}_n)^2 = \sum_{i=1}^{n}\left(X_i^2 - 2X_i\bar{X}_n + \bar{X}_n^2\right)$$

$$= \sum_{i=1}^{n} X_i^2 - 2n\bar{X}_n^2 + n\bar{X}_n^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$$

$$E[s] = E\left[\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right] = \sum_{i=1}^{n} E[X_i^2] - nE[\bar{X}^2]$$

$$= n\underbrace{E[X^2]}_{\sigma_X^2} - n\underbrace{E[\bar{X}^2]}_{\frac{\sigma_X^2}{n}}$$ variance of sample mean (if $m_X = 0$)

$$= (n-1)\sigma_X^2$$ which we already knew from Eqn 8.16

Now consider the second moment

$$(*) \quad E[s^2] = E\left[\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n} X_i^2\right)^2\right] - 2nE\left[\bar{X}^2\sum_{i=1}^{n} X_i^2\right] + n^2E[\bar{X}^4]$$

Take each of these terms separately

8.21 1st term:

$$E\left[\left(\sum_{i=1}^{n} X_i^2\right)^2\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} E[X_i^2 X_j^2]$$

$$= \sum_{i=1}^{n} E[X_i^4] + \sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n} E[X_i^2]E[X_j^2]$$

$$= nE[X^4] + n(n-1)E[X^2]^2$$

2nd term:

$$E\left[\bar{X}_n^2 \sum_{i=1}^{n} X_i^2\right] = E\left[\frac{1}{n^2}\sum_i X_i \sum_j X_j \sum_k X_k^2\right]$$

$$= \frac{1}{n^2}\sum_k E[X_k^4] + \frac{1}{n^2}\sum_{k=1}^{n} E[X_k^2] \sum_{i=j\neq k}\sum_j E[X_i^2] + 0$$

$$= \frac{1}{n}E[X^4] + \frac{n(n-1)E[X^2]^2}{n^2}$$

3rd term:

$$E[\bar{X}_n^4] = \frac{1}{n^4} E\left[\sum_i X_i \sum_j X_j \sum_k X_k \sum_l X_l\right]$$

$$= \frac{1}{n^4}\Bigg[\sum_{i=j=k=l} E[X_i^4] + \sum_{\substack{i=1 \\ j=i}}^{n} E[X_i^2] \sum_{\substack{k=l \\ k\neq i}}^{n} E[X_k^2]$$

$$+ \sum_{\substack{i=1 \\ k=l}}^{n} E[X_i^2] \sum_{\substack{j=l \\ j\neq i}} E[X_j^2]$$

$$+ \sum_{\substack{i=1 \\ l=i}}^{n} E[X_i^2] \sum_{\substack{j=k \\ j\neq i}} E[X_j^2]\Bigg]$$

$$= \frac{1}{n^4}\left[nE[X^4] + 3n(n-1)E[X^2]^2\right]$$

(8.21) substitute 3 terms back into (*)

$$E[s^2] = nE[X^4] + n(n-1)E[X^2]^2$$
$$-2n\left\{ \frac{E[X^4]}{n} + \frac{n(n-1)E[X^2]^2}{n^2} \right\}$$
$$+ \frac{n^2}{n^4}\left\{ nE[X^4] + 3(n)(n-1)E[X^2]^2 \right\}$$

$$= E[X^4]\left\{ n - 2 + \frac{1}{n} \right\}$$
$$+E[X^2]^2\left\{ n(n-1) - 2(n-1) + \frac{3(n-1)}{n} \right\}$$

$$VAR[s] = E[s^2] - E[s]^2 \qquad E[s]^2 = (n-1)^2 E[X^2]^2$$

$$= E[X^4]\left\{ \frac{n^2 - 2n + 1}{n} \right\}$$
$$+E[X^2]^2 \left\{ \underbrace{\frac{n^2(n-1) - 2n(n-1) + 3(n-1) - n(n-1)^2}{n}}_{-\frac{(n-1)(n-3)}{n}} \right\}$$

$$= E[X^4]\frac{(n-1)^2}{n} - E[X^2]^2 \frac{(n-1)(n-3)}{n}$$

$$VAR\left[\frac{1}{n-1}\sum_i (X_i - \overline{X}_n)^2\right] = VAR\left[\frac{1}{n-1}s\right] = \frac{1}{(n-1)^2}VAR[s]$$
$$= \frac{1}{n}E[X^4] - E[X^2]^2\frac{(n-3)}{n(n-1)}$$
$$= \frac{1}{n}\left[E[X^4] - \frac{n-3}{n-1}E[X^2]^2 \right]$$
$$= \frac{1}{n}\left[E[(X-m_x)^4] - \frac{n-3}{n-1}E[(X-m_x)^2]^2 \right]$$
$$= \frac{1}{n}\left[\mu_4 - \frac{n-3}{n-1}\sigma_x^4 \right]$$

8.22

```
x = normal_rnd (0, 1, 2, 2000)
y = A' * x
cxy1 = y(1,:) .* y(2,:)
z = reshape (cxy1, 20, 100)
hist (mean (z))
mean(mean(z)') = 0.50009

% for unknown means and variances
for i = 1:100
mx(i) = mean (y(1, i:i+20));
my(i) = mean (y(2, i: i+20));
xy(i) = sum( y(1, i: i+20) .* y(2, i: i+20));
cxy2 (i) = (xy(i)-20*mx(i) my(i))/19
end
hist(cxy2)
mean( cxy2') = 0.51493
```

$= \{X_i Y_i\}$

$\frac{1}{20}\sum_{i=1}^{20} X_i Y_i$, known mean

8.23

```
x= normal_rnd(0,1,2,2000);
y= A' *x

for i=1:100
mx(i) = mean(y(1, i:i+20))
my(i) = mean(y(2, i:i+20))
vx(i) = sum((y(1, i: i+20)-mx(i)).^2)
vy(i) = sum((y(2, i: i+20) -my(i)).^2)
rhoxy(i) = (xy(i) -20* mx(i) *my(i))/ sqrt(vx(i)*vy(i))
end
hist (rhoxy)
mean (rhoxy') = 0.49916
```

8.3 Maximum Likelihood Estimation

8.24

a) $f(x) = \frac{1}{\theta} e^{-x/\theta} \quad x \geq 0$

$$f(x_1, \ldots, x_n | \theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta} = \frac{1}{\theta^n} e^{-\sum_{i=1}^{n} x_i / \theta}$$

$$\ln f(x_1 \ldots x_n | \theta) = -n \ln \theta - \frac{1}{\theta} \sum_{i=1}^{n} x_i$$

$$0 = \frac{d}{d\theta} \ln f = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_i x_i \Rightarrow n\theta = \sum_i x_i$$

$$\hat{\theta}_{ML} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

b) By invariance property

$$\hat{\lambda}_{ML} = \frac{1}{\hat{\theta}_{ML}} = \frac{1}{\frac{1}{n} \sum_{i=1}^{n} X_i}$$

Try direct approach anyway:

$$f(x_1 \ldots x_n | \lambda) = \lambda^n \prod_{i=1}^{n} e^{-\lambda \sum_i x_i}$$

$$0 = \frac{d}{d\lambda} \ln f = \frac{d}{d\lambda} \left[n \ln \lambda - \lambda \sum_i X_i \right] = \frac{n}{\lambda} - \sum_i X_i$$

$$\Rightarrow \hat{\lambda} = \frac{n}{\sum_i X_i} \checkmark$$

c) $\hat{\theta}_{ML} = \frac{1}{n} \sum_{i=1}^{n} X_i$ scaled version of n Erlang RV

$$\hat{\lambda}_{ML} = \frac{1}{\hat{\theta}_{ML}} \Rightarrow f_{\lambda}(y) = \frac{f_{\theta}(1/y)}{y^2} \quad y > 0$$

where f_{θ} is n-Erlang.

d) $\hat{\theta}_{ML}$ is unbiased and consistent because it is a sample mean

8.25

(a) $$f_X(x_1,\dots,x_n|\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(x-\theta-1)^2/2}$$

$$0 = \frac{d}{d\theta}\ln f = \frac{d}{d\theta}\sum_{i=1}^{n}\left(\ln\frac{1}{\sqrt{2\pi}} - \frac{1}{2}(x_i-\theta-1)^2\right)$$

$$= +\sum_{i=1}^{n}(x_i-\theta-1) = \sum_{i=1}^{n} x_i - n\theta - n$$

$$\hat{\theta}_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i - 1$$

(b) $\hat{\theta}_{ML}$ is Gaussian with mean

$$E[\hat{\theta}_{ML}] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] - 1 = \frac{1}{n} n[\theta+1] - 1 = \theta$$

$$VAR[\hat{\theta}_{ML}] = \frac{1}{n} VAR[X] = \frac{1}{n}$$

(c) From (b) $\hat{\theta}_{ML}$ is unbiased

$VAR[\hat{\theta}_{ML}] \to 0$ so $\hat{\theta}_{ML}$ is consistent.

8.26

$$f(x_1,\dots,x_n|\theta) = \frac{1}{\theta^n} \qquad 0 \le x_i \le \theta$$

f is increases as θ decreases

$\therefore$ f is maximized when $\theta = \max(X_1,\dots,X_n)$.

8.27

a) $$f(x_1,\ldots,x_n \mid \alpha) = \prod_{i=1}^{n} \alpha \frac{x_m^{\alpha}}{x_i^{\alpha+1}} \qquad x_i \geq x_m$$

$$0 = \frac{d}{d\alpha} f = \frac{d}{d\alpha} \sum_{i=1}^{n} \left(\ln\alpha + \alpha \ln x_m - (\alpha+1)\ln x_i \right)$$

$$= \sum_{i=1}^{n} \left(\frac{1}{\alpha} + \ln x_m - \ln x_i \right) = \frac{n}{\alpha} + n \ln x_m - \sum_{i=1}^{n} \ln x_i$$

$$\frac{n}{\alpha} = -n\ln x_m + \sum_{i=1}^{n} \ln x_i$$

$$\alpha = \frac{n}{-n\ln x_m + \sum_{i=1}^{n} \ln x_i} = \frac{n}{\sum_{i=1}^{n} \ln(x_i / x_m)}$$

b) If x_m is unknown we have additional condition

$$f(x_1,\ldots,x_n \mid \alpha, x_m) = \frac{\alpha^n x_m^{n\alpha}}{x_i^{\alpha+1}} \qquad x_i \geq x_m$$

f is an increasing function of x_m, so f is maximized by letting $\underline{\underline{\hat{x}_m = \min(x_1,\ldots,x_n)}}$.

Then

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \ln(x_i / \hat{x}_m)}$$

8.27 c) Consider $1/\alpha$ first:

$$\frac{1}{\alpha} = \frac{1}{n}\sum_{i=1}^{n} \ln X_i/X_m$$

- X_i/X_m is a normalized Pareto, which has long tail
- $\ln X_i/X_m$ compacts the tail closer to the origin
- the estimator takes the arithmetic average

Let $Y = X/x_m$

$$P[Y > y] = P\left[\ln \frac{X}{x_m} > y\right] = P[X > x_m e^{y}]$$

$$= \frac{x_m^{\alpha}}{(x_m e^{y})^{\alpha}} = e^{-\alpha y} \quad \text{exponential RV with mean } 1/\alpha$$

$\therefore$ above estimator is sample mean for the transformed random variable

$\therefore$ the estimator is consistent.

$\hat{x}_m = \min(x_1, \ldots, x_n)$ has pdf given by

$$f_{\hat{X}_m}(x) = n\left[1 - F_X(x)\right]^{n-1} f_X(x) = n\left(\frac{x_m^{\alpha}}{x^{\alpha}}\right)^{n-1} \alpha \frac{x_m^{\alpha}}{x^{\alpha+1}} \qquad x \geq x_m$$

$$= n\alpha \frac{x_m^{\alpha n}}{x^{\alpha n}} \frac{1}{x} \quad \text{which is Pareto with parameter } \alpha n$$

with

$$E[\hat{X}_m] = \frac{\alpha n x_m}{\alpha n - 1} \to x_m$$

$$VAR[\hat{X}_m] = \frac{\alpha n x_m^2}{(\alpha n - 2)(\alpha n - 1)} \to 0$$

$\Rightarrow \hat{X}_m$ is consistent.

8.28 $\theta = \alpha^2$

$$f_X(x) = \frac{x}{\theta} e^{-x^2/2\theta} \qquad x \geq 0.$$

(a) $$f_X(x_1 \ldots x_n | \theta) = \prod_{i=1}^{n} \frac{x_i}{\theta} e^{-x_i^2/2\theta}$$

$$0 = \frac{d}{d\theta} \ln f = \frac{d}{d\theta} \sum_{i=1}^{n} \left(\ln x_i - \ln \theta - x_i^2/2\theta \right)$$

$$0 = \sum_{i=1}^{n} \left(-\frac{1}{\theta} + \frac{x_i^2}{2\theta^2} \right) \Rightarrow 0 = -n\theta + \frac{1}{2} \sum_{i=1}^{n} x_i^2$$

$$\hat{\theta} = \frac{1}{2n} \sum_{i=1}^{n} X_i^2$$

X_i^2 is an exponential RV with $\lambda = 1/2\alpha^2$

$$\therefore E[\hat{\theta}] = \frac{1}{2n} \sum_{i=1}^{n} E[X_i^2] = \alpha^2 \quad \text{unbiased.}$$

$$VAR[\hat{\theta}] = \frac{1}{4n} VAR[X_i] = \frac{1}{4n} (4\alpha^2)^2 = \frac{\alpha^4}{n} \to 0$$

$\therefore \hat{\theta}$ is consistent

8.29

$$f_X(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} x^{\alpha-1} = \frac{1}{\alpha} x^{\alpha-1} \quad \text{since } \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

$$0 < x < 1$$

(a)

$$f_X(x_1 \ldots x_n) = \prod_{i=1}^{n} \frac{x_i^{\alpha-1}}{\alpha}$$

$$0 = \frac{d}{d\alpha} \ln f = \frac{d}{d\alpha} \sum_{i=1}^{n} \left(-\ln\alpha + (\alpha-1)\ln x_i \right)$$

$$= \sum_{i=1}^{n} \left(-\frac{1}{\alpha} + \ln x_i \right) = -\frac{n}{\alpha} + \sum_{i=1}^{n} \ln x_i$$

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

8.30

$$f_X(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} \qquad t > 0$$

$$f_X(x_1 \ldots x_n) = (\alpha\beta)^n \prod_{i=1}^{n} x_i^{\beta-1} e^{-\alpha x_i^{\beta}}$$

$$0 = \frac{d}{d\alpha} \ln f = \frac{d}{d\alpha} \left[n \ln \alpha\beta + \sum_{i=1}^{n} \left((\beta-1)\ln x_i - \alpha x_i^{\beta} \right) \right]$$

$$= \frac{n}{\alpha} + \sum_{i=1}^{n} -x_i^{\beta}$$

$$\alpha = \frac{n}{\sum_{i=1}^{n} x_i^{\beta}}$$

8.31 $P_f = (1 - e^{-T/\tau})$ where $\tau = 1/\lambda$ is the mean lifetime

$$P[N_f = k] = \binom{n}{k} P_f^k (1-P_f)^{n-k}$$

the ML estimate for P_f is

$$\hat{P}_f = \frac{k}{n}$$

We are interested in the following function of P_f

$$e^{-T/\tau} = 1 - P_f$$

$$-T/\tau = \ln(1-P_f)$$

$$\tau = -\frac{T}{\ln(1-P_f)}$$

By invariance property

$$\hat{\tau}_{ML} = -\frac{T}{\ln(1-\frac{k}{n})}$$

8.32 $$f_X(x) = \frac{\lambda(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x} \qquad x > 0 \quad \alpha > 0 \; \lambda > 0$$

a) $$f_X(x_1, \ldots, x_n | \lambda) = \prod_{i=1}^{n} \frac{\lambda(\lambda x_i)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x_i}$$

$$0 = \frac{d}{d\lambda} \ln f = \frac{d}{d\lambda}\left[\sum_{i=1}^{n} \left(\alpha \ln\lambda + (\alpha-1)\ln x_i - \lambda x_i - \ln\Gamma(\alpha) \right)\right]$$

$$0 = \sum_{i=1}^{n} \left(\frac{\alpha}{\lambda} - x_i \right) = \frac{n\alpha}{\lambda} - \sum_{i=1}^{n} x_i$$

$$\hat{\lambda} = \frac{n\alpha}{\sum_{i=1}^{n} x_i}$$ reciprocal of sample mean scaled by α.

b) also require

$$0 = \frac{d}{d\alpha} \ln f = \sum_{i=1}^{n} \left[\ln\lambda + \ln x_i - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} \right]$$

$$= n \ln\lambda + \sum_{i=1}^{n} \ln x_i - n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$$

We replace λ by $\hat{\lambda}$ from part a)

$$0 = n \ln\left(\frac{\alpha}{\overline{X}_n}\right) + \sum_{i=1}^{n} \ln x_i - n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$$

$$\underbrace{\ln\alpha - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}}_{} = \ln \overline{X}_n - \frac{1}{n} \sum_{i=1}^{n} \ln x_i$$

must be solved for α.

8.33 First assume known $m_X=0$, $m_Y=0$ $\sigma_X^2=1$, $\sigma_Y^2=1$

(a) Find ML estimate for ρ:

$$f(x,y) = \frac{\exp\left\{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\right\}}{2\pi(1-\rho^2)^{1/2}}$$

$$\ln \prod_{i=1}^{n} f(x_i,y_i) = \sum_{i=1}^{n}\left(-\ln 2\pi - \frac{1}{2}\log(1-\rho^2) - \frac{x_i^2-2\rho x_i y_i+y_i^2}{2(1-\rho^2)}\right)$$

$$= -n\ln 2\pi - \frac{n}{2}\ln(1-\rho^2) - \frac{1}{2(1-\rho^2)}\sum_{i=1}^{n}\left(x_i^2-2\rho x_i y_i+y_i^2\right)$$

$$0 = \frac{\partial}{\partial\rho}(\ \ ) = \frac{n\rho}{1-\rho^2} - \frac{\cancel{2}\rho}{\cancel{2}(1-\rho^2)^2}\sum(\ \ ) + \frac{2}{2(1-\rho^2)}\sum x_i y_i$$

multiply by $(1-\rho^2)^2$

$$0 = n\rho(1-\rho^2) - \rho\sum(x_i^2-2\rho x_i y_i+y_i^2) + (1-\rho^2)\sum x_i y_i$$

$$= n\rho - n\rho^3 - \rho\sum x_i^2 + 2\rho^2\sum x_i y_i - \rho\sum y_i^2 + (1-\rho^2)\sum x_i y_i$$

$$0 = \rho - \rho^3 + (1+\rho^2)\frac{1}{n}\sum x_i y_i - \rho\left(\frac{1}{n}\sum x_i^2 + \frac{1}{n}\sum y_i^2\right)$$

cubic eqn in ρ.

There is always at least one root in $-1<\rho<1$

If more than one root pick the root that gives the maximum likelihood.

8.33 – contd –

$$0=\frac{\partial \ln f}{\partial \rho}=\frac{1}{(1-\rho^2)}\left\{n\rho-\frac{1}{1-\rho^2}\left[\rho\,\frac{\sum_i (X_i-m_X)^2}{\sigma_X^2}+\rho\,\frac{\sum_i (Y_i-m_Y)^2}{\sigma_Y^2}\right.\right.$$

$$\left.\left.-(1+\rho^2)\,\frac{\sum_i (X_i-m_X)(Y_i-m_Y)}{\sigma_X\sigma_Y}\right]\right\} \quad (v)$$

We have 5 equations for the 5 unknown parameters.

(iii) and (iv) become

$$(iii')\qquad n(1-\rho^2)=\frac{\sum_i (X_i-m_X)^2}{\sigma_X^2}-\rho\,\frac{\sum_i (X_i-m_X)(Y_i-m_Y)}{\sigma_X\sigma_Y}$$

$$(iv')\qquad n(1-\rho^2)=\frac{\sum_i (Y_i-m_Y)^2}{\sigma_Y^2}-\rho\,\frac{\sum_i (X_i-m_X)(Y_i-m_Y)}{\sigma_X\sigma_Y}$$

(v) becomes

$$n(1-\rho^2)=\frac{\sum_i (X_i-m_X)^2}{\sigma_X^2}+\frac{\sum_i (Y_i-m_Y)^2}{\sigma_Y^2}-\frac{1+\rho^2}{\rho}\,\frac{\sum_i (X_i-m_X)(Y_i-m_Y)}{\sigma_X\sigma_Y}$$

subtract (iii) + (iv) from (v):

$$-n(1-\rho^2)=\underbrace{-\frac{1+\rho^2-\rho^2-\rho^2}{\rho}}_{-\frac{1-\rho^2}{\rho}}\,\frac{\sum_i (X_i-m_X)(Y_i-m_Y)}{\sigma_X\sigma_Y}$$

$$\boxed{\Rightarrow\ \hat{\rho}=\frac{\frac{1}{n}\sum_i (X_i-m_X)(Y-m_Y)}{\sigma_X\sigma_Y}}$$

8.33

(b) $$f(x,y) = \frac{\exp\left\{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-m_x}{\sigma_x}\right)^2 + 2\rho\left(\frac{x-m_x}{\sigma_x}\right)\left(\frac{y-m_y}{\sigma_y}\right) + \left(\frac{y-m_y}{\sigma_y}\right)^2\right\}\right\}}{2\pi\sqrt{\sigma_x^2\sigma_y^2(1-\rho^2)}}$$

$$\ln \prod_i f(x_i,y_i) = -n\ln 2\pi - \frac{n}{2}\left(\ln\sigma_x^2 + \ln\sigma_y^2 + \ln(1-\rho^2)\right) - \frac{1}{2(1-\rho^2)}\sum_{i=1}^{n}\left\{\left(\frac{x_i-m_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-m_x}{\sigma_x}\right)\left(\frac{y-m_y}{\sigma_y}\right) + \left(\frac{y_i-m_y}{\sigma_y}\right)^2\right\}$$

We take derivatives wrt to m_x, m_y, σ_x^2, σ_y^2, and ρ

$$0 = \frac{\partial \ln\prod f}{\partial m_x} = \frac{-1}{2(1-\rho^2)}\sum_{i=1}^{n}\left(-2\left(\frac{x_i-m_x}{\sigma_x^2}\right) + \frac{2\rho}{\sigma_x}\left(\frac{y_i-m_y}{\sigma_y}\right)\right)$$

$$= \frac{n}{\sigma_x(1-\rho^2)}\left[\frac{\frac{1}{n}\sum_i x_i - m_x}{\sigma_x} - \rho\,\frac{\frac{1}{n}\sum_i y_i - m_y}{\sigma_y}\right] \qquad (i)$$

for m_y we have

$$0 = \frac{n}{\sigma_y(1-\rho^2)}\left[\frac{\frac{1}{n}\sum_i y_i - m_y}{\sigma_y} - \rho\,\frac{\frac{1}{n}\sum_i x_i - m_x}{\sigma_x}\right] \qquad (ii)$$

$$0 = \frac{\partial \ln\prod f}{\partial \sigma_x^2} = -\frac{n/2}{\sigma_x^2} - \frac{1}{2(1-\rho^2)}\sum_{i=1}^{n}\left(-\frac{(x_i-m_x)^2}{\sigma_x^4} - 2\rho\frac{(x_i-m_x)}{\sigma_x^3}\left(\frac{y_i-m_y}{\sigma_y}\right)\left(-\frac{1}{2}\right)\right)$$

(iii) $$= -\frac{1}{2\sigma_x^2(1-\rho^2)}\left[n(1-\rho^2) - \frac{\sum_i(x_i-m_x)^2}{\sigma_x^2} + \rho\,\frac{\sum_i(x_i-m_x)(y_i-m_y)}{\sigma_x\sigma_y}\right]$$

Similarly

(iv) $$0 = -\frac{1}{2\sigma_y^2(1-\rho^2)}\left[n(1-\rho^2) - \frac{\sum_i(y_i-m_y)^2}{\sigma_y^2} + \rho\,\frac{\sum_i(x_i-m_x)(y_i-m_y)}{\sigma_x\sigma_y}\right]$$

8.33 substitute ρ into (iii')

$$n(1-\rho^2) = \frac{\sum_i (x_i - m_x)^2}{\sigma_X^2} - n\rho^2$$

$$\Rightarrow \boxed{\hat{\sigma}_X^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - m_x)^2}$$

Similarly we obtain

$$\boxed{\hat{\sigma}_Y^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - m_Y)^2}$$

Finally we obtain the estimates for m_X and m_Y from (i) and (ii)

(i') $$\frac{\overline{X}_n - m_X}{\sigma_X} = \rho\,\frac{\overline{Y}_n - m_Y}{\sigma_Y}$$

(ii') $$\frac{\overline{Y}_n - m_Y}{\sigma_Y} = \rho\,\frac{\overline{X}_n - m_X}{\sigma_X} = \rho^2\,\frac{\overline{Y}_n - m_Y}{\sigma_Y^2}$$

$$\Rightarrow \overline{Y}_n - m_Y = 0 \Rightarrow \boxed{\hat{m}_Y = \frac{1}{n}\sum_i Y_i}$$

Similarly $$\boxed{\hat{m}_X = \frac{1}{n}\sum X_i}.$$

8.34 Invariance Property

ML estimator for $h(\theta)$ finds h^* such that

$$f(x_1 \ldots x_n | h^*) = \max f(x_1 \ldots x_n | h^*)$$

ML estimator for θ finds θ^* such that

$$f(x_1 \ldots x_n | \theta^*) = \max f(x_1 \ldots x_n | \theta^*)$$

Let $\theta_0 = h^{-1}(h^*)$ the inverse image of the optimum h^* and suppose that $\theta_0 \neq \theta^*$ the optimal MLE for θ, then

$$f(x_1 \ldots x_n | \theta^*) = f(x_1 \ldots x_n | h(\theta^*))$$
$$\leq f(x_1 \ldots x_n | h^*) = f(x_1 \ldots x_n | \theta_0)$$

contradicting the optimality of θ^*.

8.35 From (8.35) we have

$$0 = E\left[\frac{\partial}{\partial\theta}\ln f_X(\underline{x}|\theta)\right] = \int\cdots\int_{\underline{x}_n}\left(\frac{\partial}{\partial\theta}\ln f_X(\underline{x}|\theta)\right) f_X(\underline{x}|\theta)\, d\underline{x}$$

Take another derivative wrt θ

$$0 = E\left[\frac{\partial^2}{\partial\theta^2}\ln f_X(\underline{x}|\theta)\right]$$

$$= \iint_{\underline{x}_n}\left\{\frac{\partial^2}{\partial\theta^2}\ln f_X(\underline{x}|\theta)\, f_X(\underline{x}|\theta) + \left(\frac{\partial}{\partial\theta}\ln f_X(\underline{x}|\theta)\right)\frac{\partial}{\partial\theta}f_X(\underline{x}|\theta)\right\} d\underline{x}$$

Note that $\dfrac{\partial f(\underline{x}|\theta)}{\partial\theta} = \left(\dfrac{\partial}{\partial\theta}\ln f(\underline{x}|\theta)\right) f(\underline{x}|\theta)$, so

$$0 = E\left[\frac{\partial^2}{\partial\theta^2}\ln f(\underline{x}|\theta)\right] + \iint_{\underline{x}_n}\left(\frac{\partial}{\partial\theta}\ln f_X(\underline{x}|\theta)\right)^2 f(\underline{x}|\theta)\, d\underline{x}$$

$$\Rightarrow \quad E\left[\frac{\partial^2}{\partial\theta^2}\ln f(\underline{x}|\theta)\right] = -E\left[\left(\frac{\partial}{\partial\theta}\ln f(\underline{x}|\theta)\right)^2\right]$$

$$= -I_n(\theta).$$

8.36

(a) Binomial

$$\ln l(\underline{X}|p) = \sum_{i=1}^{n} \left(\ln\binom{n}{k_i} + k_i \ln p + (n-k_i)\ln(1-p) \right)$$

$$\frac{\partial}{\partial p}\ln l(\underline{X}|p) = \sum_{i=1}^{n} \left(\frac{k_i}{p} - \frac{n-k_i}{1-p} \right)$$

$$\frac{\partial^2}{\partial p}\ln l(\underline{X}|p) = \sum_{i=1}^{n} \left(-\frac{k_i}{p^2} - \frac{n-k_i}{(1-p)^2} \right)$$

$$-E\left[\frac{\partial^2}{\partial p}\ln l(\underline{X}|p)\right] = + \sum_{i=1}^{n} \left(\frac{E[k_i]}{p^2} + \sum_{i=1}^{n} \frac{n-E[k_i]}{(1-p)^2} \right.$$

$$= \frac{n^2}{p} + \frac{n^2}{1-p} = \frac{n^2}{p(1-p)}$$

8.36b Gaussian: known σ^2 unknown mean:

$$\ln l(\underline{X}|\mu) = \sum_{i=1}^{n} \left(\ln\sqrt{2\pi}\,\sigma^2 \mp (x_i-\mu)^2/2\sigma^2 \right)$$

$$\frac{\partial}{\partial\mu}\ln l = \sum_{i=1}^{n} + 2(x_i-\mu)/2\sigma^2$$

$$\frac{\partial^2}{\partial\mu}\ln l = \sum_{i=1}^{n} -2/2\sigma^2 = -\frac{n}{\sigma^2}$$

$$I_n(\mu) = -E\left[\frac{\partial^2}{\partial\mu}\ln l\right] = \frac{n}{\sigma^2}.$$

8.36 c Gaussian unknown variance, known mean

$$\ln l(\underline{X}|\sigma^2) = \sum_{i=1}^{n}\left(-\ln\sqrt{2\pi\sigma^2} - (x_i-\mu)^2/2\sigma^2\right)$$

$$\frac{\partial}{\partial\sigma^2}\ln l(\underline{X}|\sigma^2) = \sum_{i=1}^{n}\left(\frac{-1}{2\sigma^2} + \frac{(x_i-\mu)^2}{2(\sigma^2)^2}\right)$$

$$\frac{\partial^2}{\partial\sigma^2}\ln l(\underline{X}|\sigma^2) = \sum_{i=1}^{n}\left(\frac{+1}{2(\sigma^2)^2} + \frac{(x_i-\mu)^2}{2(\sigma^2)^3}(-2)\right)$$

$$I_n(\sigma^2) = \frac{-n}{2(\sigma^2)^2} + \frac{1}{(\sigma^2)^3}\underbrace{\sum_{i=1}^{n}E[(X_i-\mu)^2]}_{n\sigma^2}$$

$$= -\frac{n}{2\sigma^4} + \frac{n}{\sigma^4} = \frac{n}{2\sigma^4}$$

If the mean μ is unknown, the above computation does not change.

From Ex 8.8 the variance of the unbiased sample variance estimator is

$$VAR[\hat{\sigma}_n^2] = \frac{1}{n}\left[\mu_4 - \frac{n-3}{n-1}\sigma^4\right] \overset{\text{Gaussian}}{=} \frac{1}{n}\left[3\sigma^4 - \frac{n-3}{n-1}\sigma^4\right]$$

$$= \frac{2\sigma^4}{n-1} > \frac{2\sigma^4}{n} = \text{Cramer Rao LB.}$$

8.36d $\quad f_X(x|\beta) = \dfrac{x^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha)} e^{-x/\beta}$

(a) $$\ln l(\underline{x}|\beta) = \sum_{i=1}^{n}\left(-\alpha\ln\beta - \ln\Gamma(\alpha) + (\alpha-1)\ln x_i - \frac{x_i}{\beta}\right)$$

$$\frac{\partial}{\partial\beta}\ln l = \sum_{i=1}^{n}\left(-\frac{\alpha}{\beta} + \frac{x_i}{\beta^2}\right)$$

$$\frac{\partial^2}{\partial\beta}\ln l = \sum_i\left(\frac{\alpha}{\beta^2} - \frac{2x_i}{\beta^3}\right)$$

$$I(\beta) = -\left(\frac{n\alpha}{\beta^2} - \frac{2}{\beta^3}\underbrace{\sum_i E[X]}_{n\alpha\beta}\right) = -\left(\frac{n\alpha}{\beta^2} - \frac{2n\alpha}{\beta^2}\right) = +\frac{n\alpha}{\beta^2}$$

8.36e

$$\ln l = \sum_i\left(k_i\ln\alpha \mp \ln k_i! - \alpha\right)$$

$$\frac{\partial}{\partial\alpha}\ln l = \sum_i \frac{k_i}{\alpha} - 1$$

$$\frac{\partial^2}{\partial\alpha^2}\ln l = \sum_i -\frac{k_i}{\alpha^2}$$

$$I_n(\alpha) = +E\left[\sum_i\frac{k_i}{\alpha^2}\right] = \frac{1}{\alpha^2}n\alpha = \frac{n}{\alpha}.$$

8.37 $\hat{\theta}_{ML}$ estimator for $1/\lambda$, the mean of an exponential RV

from 8.24a $\hat{\theta}_{ML} = \frac{1}{n}\sum_i X_i$

from invariance property $\hat{\theta}^2_{ML}$ is the ML estimator for $1/\lambda^2$

We are interested in

$$P\left[-\frac{1}{20\lambda^2} < \hat{\theta}^2_{ML} - \frac{1}{\lambda^2} < \frac{1}{20\lambda^2}\right]$$

$$= P\left[\frac{19}{20\lambda^2} < \hat{\theta}^2_{ML} < \frac{21}{20\lambda^2}\right]$$

$$= P\left[\sqrt{\frac{19}{20\lambda^2}} < \hat{\theta}_{ML} < \sqrt{\frac{21}{20\lambda^2}}\right]$$

$\hat{\theta}_{ML}$ has mean $\frac{1}{\lambda}$ and variance $\frac{1}{n\lambda^2}$ and is approx. Gaussian

$$\simeq \int_{\sqrt{\frac{19}{20\lambda^2}}}^{\sqrt{\frac{21}{20\lambda^2}}} \frac{1}{\sqrt{2\pi\left(\frac{1}{n\lambda^2}\right)}}\, e^{-\left(x-\frac{1}{\lambda}\right)/2\left(\frac{1}{n\lambda^2}\right)}\, dx$$

$$= Q\left(\frac{\sqrt{\frac{19}{20\lambda^2}} + \frac{1}{\lambda}}{\frac{1}{\sqrt{n\lambda^2}}}\right) - Q\left(\frac{\sqrt{\frac{21}{20\lambda^2}} + 1}{\frac{1}{\sqrt{n\lambda^2}}}\right)$$

$$= Q\left(\frac{\sqrt{\frac{19}{20}} + 1}{1/\sqrt{n}}\right) - Q\left(\frac{\sqrt{\frac{21}{20}} + 1}{1/\sqrt{n}}\right).$$

8.38 $\hat{\theta}_{ML} = \frac{1}{n}\sum_{i=1}^{n} X_i$ estimator for α, Poisson

Estimate $\hat{h}(\theta) = e^{-\hat{\theta}_{ML}}$ estimator for $P[N=0]$.

$\hat{h}(\theta)$ is ML est for $P[N=0]$ by invariance property

We are interested in

$$P\left[\left|e^{-\hat{\theta}} - e^{-\alpha}\right| < \frac{1}{10}e^{-\alpha}\right]$$

$$= P\left[-\frac{1}{10}e^{-\alpha} < e^{-\hat{\theta}} - e^{-\alpha} < \frac{1}{10}e^{-\alpha}\right]$$

$$= P\left[\frac{9}{10}e^{-\alpha} < e^{-\hat{\theta}} < \frac{11}{10}e^{-\alpha}\right]$$

$$= P\left[\alpha + \ln\frac{10}{11} < \hat{\theta} < \alpha + \ln\frac{10}{9}\right]$$

[$\hat{\theta}_{ML}$ has mean α and variance $\frac{\alpha}{n}$ and is approx Gaussian]

$$\approx \int_{\alpha+\ln\frac{10}{11}}^{\alpha+\ln\frac{10}{9}} \frac{1}{\sqrt{2\pi\alpha/n}}\, e^{-(x-\alpha)/2(\alpha/n)}\, dx$$

$$= Q\left(\frac{\ln\frac{10}{11}}{\alpha/n}\right) - Q\left(\frac{\ln\frac{10}{9}}{\alpha/n}\right)$$

we have dependence on the actual value of α

8.4 Confidence Intervals

8.39

The ith measurement is $X_i = m + N_i$ where $\mathcal{E}[N_i] = 0$ and $VAR[N_i] = 10$. The sample mean is $M_{100} = 100$ and the variance is $\sigma = \sqrt{10}$.

Eqn. ~~5.37~~ 8.52 with $z_{\alpha/2} = 1.96$ gives

$$\left(100 - \frac{1.96\sqrt{10}}{\sqrt{30}}, 100 + \frac{1.96\sqrt{10}}{\sqrt{30}}\right) = (98.9, 101.1)$$

8.40

5.32 The width of the confidence interval given by Eqn. ~~5.37~~ 8.52 is

$$\left(M_n + \frac{z_{\alpha/2}\sigma}{\sqrt{n}}\right) - \left(M_n - \frac{z_{\alpha/2}\sigma}{\sqrt{n}}\right) = \frac{2z_{\alpha/2}\sigma}{\sqrt{n}}$$

a) For 95% confidence intervals $z_{\alpha/2} = 1.96$, so $(\sigma = 1)$

$$\text{width of interval} = \frac{2(1.96)}{\sqrt{n}} = \begin{cases} 1.96 & n = 4 \\ 0.98 & n = 16 \\ 0.29 & n = 100 \end{cases}$$

b) For 99% confidence intervals $z_{\alpha/2} = 2.576$ so

$$\text{width of interval} = \frac{2(2.576)}{\sqrt{n}} = \begin{cases} 2.576 & n = 4 \\ 1.288 & n = 16 \\ 0.515 & n = 100 \end{cases}$$

8.41

5.33 $M_n = 223 \qquad V_N^2 = 100 \qquad n = 225$

$\Rightarrow V_n = 10$

(a) Assuming that individual lifetimes are Gaussian RV's, Eqn. ~~5.43~~ 8.58 with $n = \infty$

$$\begin{aligned}\left(M_n - \frac{z_{\alpha/2,\infty}V_n}{\sqrt{n}}, \frac{M_n + z_{\alpha/2,\infty}V_n}{\sqrt{n}}\right) &= \left(223 - \frac{1.96(10)}{\sqrt{225}}, 223 + \frac{1.96(10)}{\sqrt{225}}\right) \\ &= (222, 224)\end{aligned}$$

41) (b)

from Eqn 8.59 the confidence interval for the sample variance is

$$\left[\frac{224(100)}{\chi^2_{.025,224}}, \frac{224(100)}{\chi^2_{0.975,224}}\right] = \left[\frac{224(100)}{267.35}, \frac{224(100)}{184.44}\right]$$

$$= [83.785, 121.45].$$

8.42

~~5.34~~ $M_n = \frac{1}{n}\sum_{j=1}^{10} X_j = \frac{350}{10} = 35$

$$\begin{aligned}
\sum_{j=1}^{n}(X_j - M_n)^2 &= \sum_{j=1}^{n} X_j^2 - 2M_n\sum_{j=1}^{n} X_j + nM_n^2 \\
&= \sum_{j=1}^{n} X_j^2 - nM_n^2 \\
V_n^2 &= \frac{1}{n-1}\sum_{j=1}^{n}(X_j - M_n)^2 = \frac{1}{n-1}\sum_{j=1}^{n} X_j^2 - \frac{n}{n-1}M_n^2 \\
&= \frac{1}{9}(12645) - \frac{10}{9}(35)^2 = 43.88 \\
\Rightarrow V_n &= 6.624
\end{aligned}$$

For 90% confidence interval

$$z_{\alpha/2,9} = 1.833$$

So Eqn. ~~5.43~~ 8.58 gives

$$\left(35 - \frac{1.833(6.624)}{\sqrt{10}}, 35 + \frac{1.833(6.624)}{\sqrt{10}}\right) = (31.16, 38.84)$$

8.42 (b)

from Eq. 8.59:

$$\left[\frac{9(43.88)}{\underbrace{\chi^2_{0.05,9}}_{16.92}}, \frac{9(43.88)}{\underbrace{\chi^2_{0.95,9}}_{3.325}}\right] = [23.34, 118.77]$$

8.43

~~5.35~~ a) $M_n = 57.3 \qquad V_n^2 = 23.2 \qquad n = 10$

$$\left(M_n - \frac{1.833V_n}{\sqrt{10}}, M_n + \frac{1.833V_n}{\sqrt{10}}\right) = (54.5, 60.1) \qquad 90\%$$

$$\left(M_n - \frac{2.262V_n}{\sqrt{10}}, M_n + \frac{2.262V_n}{\sqrt{10}}\right) = (53.85, 60.75) \qquad 95\% \;\; \cancel{90\%}$$

$$\left(M_n - \frac{3.25V_n}{\sqrt{10}}, M_n + \frac{3.25V_n}{\sqrt{10}}\right) = (52.35, 62.25) \qquad 99\%$$

b) $M_n = 57.3 \qquad V_n^2 = 23.2 \qquad n = 20$

$$\left(M_n - \frac{1.725V_n}{\sqrt{20}}, M_n + \frac{1.725V_n}{\sqrt{20}}\right) = (55.44, 59.16) \qquad 90\%$$

$$\left(M_n - \frac{2.086V_n}{\sqrt{20}}, M_n + \frac{2.086V_n}{\sqrt{20}}\right) = (55.05, 59.55) \qquad 95\% \;\; \cancel{90\%}$$

$$\left(M_n - \frac{2.895V_n}{\sqrt{20}}, M_n + \frac{2.895V_n}{\sqrt{20}}\right) = (54.24, 60.36) \qquad 99\%$$

Note: the entry for $z_{\alpha/2,20}$ was used instead of $z_{\alpha/2,19}$.

8.43 c) $\left[\frac{9(23.2)}{\chi^2_{\alpha/2,9}}, \frac{9(23.2)}{\chi^2_{1-\alpha/2,9}}\right]$

$n = 10$ measurements

[12.34, 62.78] 90%

[10.98, 77.33] 95%

[8.85, 120.69] 99%.

$\left[\frac{19(23.2)}{\chi^2_{\alpha/2,19}}, \frac{19(23.2)}{\chi^2_{1-\alpha/2,19}}\right]$.

[14.62, 43.56] 90%

[13.42, 49.47] 95%

[11.43, 64.44] 99%.

8.44

$$\begin{aligned} M_{15} &= -1.154 \\ V_{15}^2 &= 3.711 \end{aligned}$$

8.2

From Table 5.2 with $1-\alpha = 90\%$ and $n-1 = 14$, we have

$$z_{\alpha/2,14} \approx z_{\alpha/2,15} = 1.753, \qquad \text{so}$$

$$\left(M_{15} - \frac{z_{\alpha/2,15}V_n}{\sqrt{15}}, M_{15} + \frac{z_{\alpha/2,15}V_n}{\sqrt{15}} \right) = (-2.026, -0.282)$$

8.45

~~5.37~~ The sample mean and variance of the batch sample means are $M_{10} = 24.9$ and $V_{10}^2 = 3.42$. The mean number of heads in a batch is $\mu = \mathcal{E}[M_{10}] = \mathcal{E}[X] = 50p$.
From Table ~~5~~8.2, with $1-\alpha = 95\%$ and $n-1 = 9$ we have

$$z_{\alpha/2,9} = 2.262$$

The confidence interval for μ is

$$\left(M_{10} - \frac{z_{\alpha/2,9}V_{10}}{\sqrt{10}}, M_{10} + \frac{z_{\alpha/2}V_{10}}{\sqrt{10}} \right) = (23.58, 26.22)$$

The confidence interval for $p = M_{10}/50$ is then

$$\left(\frac{23.58}{50}, \frac{26.22}{50} \right) = (0.4716, 0.5244)$$

8.46

$z = 1.645$ $n = 10$ $\sigma^2 = 1$

a) $\left(M_n - \frac{1.645\sigma}{\sqrt{10}}, M_n + \frac{1.645\sigma}{\sqrt{10}} \right) = (M_n - 0.5202, M_n + 0.5202)$

8.47 90% confidence intervals

ⓐ $n = 4$ batches $\left(M_n - \frac{2.353 V_n}{\sqrt{4}}, M_n + \frac{2.353 V_n}{\sqrt{4}}\right)$

$n = 8$ $\left(M_n - \frac{1.895 V_n}{\sqrt{8}}, M_n + \frac{1.895 V_n}{\sqrt{8}}\right)$

$n = 16$ $\left(M_n - \frac{1.753 V_n}{\sqrt{16}}, M_n + \frac{1.753 V_n}{\sqrt{16}}\right)$

$n = 32$ $\left(M_n - \frac{1.697 V_n}{\sqrt{31}}, M_n + \frac{1.697 V_n}{\sqrt{31}}\right)$

8.48 $\mu = 25$ $\sigma^2 = 36$

Gaussian mean and variance unknown.

$$\left(M_n - z_{\alpha/2, n-1}\frac{\hat{\sigma}_n}{\sqrt{n}}, M_n + z_{\alpha/2, n-1}\frac{\hat{\sigma}_n}{\sqrt{n}}\right)$$ $\alpha = 0.10$, $n = 50, 100, \ldots$

$$\left(\frac{(n-1)\hat{\sigma}_n^2}{\chi^2_{\alpha/2, n-1}}, \frac{(n-1)\hat{\sigma}_n^2}{\chi^2_{1-\alpha/2, n-1}}\right)$$ $\alpha = 0.10$, $n = 50, 100, \ldots$

8.5 Hypothesis Testing

8.49 $H_0 : \alpha = 30$ $\quad$ $n = 8$ measurements

$H_1 : \alpha > 30$ $\quad$ $\bar{X}_8 = 32 \Rightarrow \sum_{i=1}^{8} N_i = 256$

The experiments involves n measurements of a Poisson random variable. We take the sum of the total number of orders $N = \sum_{i=1}^{8} N_i$ (equivalent to taking the sample mean)

Accept H_0 if $N_T < T$ $\quad$ N Poisson with mean $n\alpha = 8\alpha$

Reject H_0 if $N_T \geq T$.

$$\alpha = 5\% = P[\text{Reject } H_0 \mid H_0] = P[N_T \geq T \mid H_0]$$

$$= \sum_{k=T}^{\infty} \frac{240^k}{k!} e^{-240} \qquad \bar{X}_8 = \frac{1}{8} N$$

$$\approx P\left[\frac{\bar{X}_8 - 30}{\sqrt{30}/\sqrt{8}} > \frac{T-30}{\sqrt{30}/\sqrt{8}}\right] = Q(1.64)$$

$$\Rightarrow T - 30 = \frac{1.64\sqrt{8}}{\sqrt{30}} + 30 = 30.847$$

$\bar{X}_8 = 32 > 30.847 \Rightarrow$ Reject H_0

$\alpha = 1\%$ $\quad$ $1\% = Q(2.326)$

$$\Rightarrow T = 30 + \frac{2.326\sqrt{8}}{\sqrt{30}} = 31.201$$

$\bar{X}_8 = 32 > 31.2 \Rightarrow$ Reject H_0

8.50

Michael \ Carlos	T	H
T	tie	C wins
H	M wins	tie

In a fair game
$\frac{1}{2}$ games are ties
$\frac{1}{4}$ Carlos wins
$\frac{1}{4}$ Michael wins

(a) If we count how many times Carlos wins N_C we are testing

$H_0: p = 1/4$ so Accept H_0 if $N_C < T$
$H_1: p > 1/4$ Reject H_0 if $N_C \geq T$

$$\alpha = 10\% = P[N_C \geq T / H_0] = \sum_{k=T}^{6} \binom{6}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{6-k}$$

for $T = 3$ $P[N_C \geq 3 | H_0] = 0.169$
$T = 4$ $P[N_C \geq 4 | H_0] = 0.0375$ Use $T = 4$ & much more stringent than 10%

for $n = 12$
$T = 5$ $P[N_C \geq 5 / H_0] = 0.157$
$T = 6$ $P[N_C \geq 6 / H_0] = 0.054$ Use $T = 6$

(b) If we count how many times Carlos' toss is heads, N_C'

$H_0: p = \frac{1}{2}$ $\alpha = 10\% = \sum_{k=T}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^n$
$H_1: p > \frac{1}{2}$

for $n = 6$ $P[N_C' \geq 5 | H_0] = 0.109$

$n = 12$ $P[N_C' \geq 9 | H_0] = 0.073$

Counting heads rather than wins is more effective because it uses more information about the expt.

8.50 (c) $P_{\text{Detection}} = P[H_1 \mid H_1]$

	1/4	3/4	
	1/8	3/8	1/2
	1/8	3/8	1/2

for $n = 6$, $p = 0.75$, counting wins

$$P_D = \sum_{k=4}^{6} \binom{6}{k} \left(\frac{3}{8}\right)^k \left(\frac{5}{8}\right)^{6-k} = 0.146 \approx 15\%$$

$n=6$ $p=0.55$ counting wins $P_D = .052 \approx 5\%$

$n=12$ $p=0.75$ counting wins $P_D = .271 \approx 21\%$

$n=12$ $p=0.55$ " $P_D = 0.082 \approx 8\%$

$\Rightarrow$ Difficult to detect by counting wins

$n=6$ $p=0.75$ counting heads $P_D = 534 \approx 53\%$

$p=0.55$ " $P_D = 0.164 \approx 16\%$

$n=12$ $p=0.75$ " $P_D = 0.649 \approx 65\%$

$p=0.55$ " $P_D = 0.134 \approx 13\%$

These results confirm that counting heads is more effective.

8.51 Gaussian $m=0 \quad \sigma^2=4$

(a) $H_0: m=0$ Accept if $-c<\bar{X}_n<c$

$H_1: m\neq 0$ Reject otherwise

$$\alpha = 0.01 = P[\,|\bar{X}_n|>c\,|\,H_0] = P\left[\left|\frac{\bar{X}_n}{2/\sqrt{n}}\right| > \frac{c}{2/\sqrt{n}}\right] = 2Q(2.576)$$

$$c = 2.576(2)/\sqrt{n} = 5.152/\sqrt{n} = 1.692$$

(b) $\bar{X}_n = -0.75 \quad n=10 \quad c=1.692$

$$|\bar{X}_{10}| = 0.75 < 1.692 \Rightarrow \text{Accept } H_0$$

(c) $$P[\text{Type II}] = P[\text{Accept } H_0 \mid H_1] = P[\,|\bar{X}_n|<c \mid H_1]$$

$$= \frac{1}{\sqrt{2\pi 4/n}} \int_{-c}^{c} e^{-(x-v)^2/2(4/n)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{-c-v}{2\sqrt{n}}}^{\frac{(c-v)}{2/\sqrt{n}}} e^{-x'^2/2}\, dx' = Q\left(\frac{-c-v}{2\sqrt{n}}\right) - Q\left(\frac{c-v}{2/\sqrt{n}}\right)$$

for $v=1,\ n=10,\ c=1.692$

$$P[\text{Type II}] = Q\left(\frac{-2.692}{2/\sqrt{10}}\right) - Q\left(\frac{0.692}{2/\sqrt{10}}\right) = .863$$

for $v = 0.01 \quad n=10 \quad c=1.692$

$$P[\text{Type II}] = Q\left(\frac{-1.702}{2/\sqrt{10}}\right) - Q\left(\frac{1.682}{2/\sqrt{10}}\right) = 0.993$$

Type II errors are very high because most samples fall in acceptance region when $v=1$ and $v=0.01$.

8.52 a) α is used to determine the most extreme value that defines the boundary of the acceptance region. α is also the probability of a sample falling in the acceptance region given H_0.

The p-value is the prob. of observing a sample as extreme or more extreme than the given observation. When the p-value equals α then the observation is at the boundary of the acceptance region.

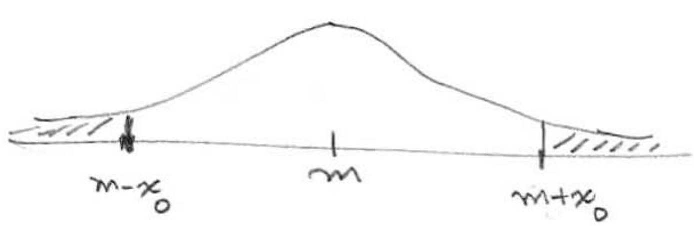
b) The hypothesis test gives a binary answer as to whether the sample is in the acceptance region or not.

The p-value gives an indication of the whether the observation would be accepted at different significance levels.

c) $P[\bar{X}_n > x_0 \mid H_0] = Q\left(\frac{x_0 - m}{\sigma/\sqrt{n}}\right)$ where x_0 is observed value

d) $P\left[\,|\bar{X}_n - m| > |x_0 - m| \;\middle|\; H_0\right]$

$$= 2\,Q\left(\frac{|x_0 - m|}{\sigma/\sqrt{n}}\right)$$



8.53

$\beta > \alpha$ Poisson rate

$H_0: \alpha = 2$ $\qquad P[X=k|H_0] = \frac{\alpha^k}{k!}e^{-\alpha}$

$H_1: \beta = 6$ $\qquad P[X=k|H_1] = \frac{\beta^k}{k!}e^{-\beta}$

(a)

$$R^c = \left\{ \underline{x} : \ln \frac{p(\underline{x}|H_1)}{p(\underline{x}|H_0)} > t \right\}$$

$$\ln \frac{p(\underline{x}|H_1)}{p(\underline{x}|H_0)} = k\ln\beta - \beta - \ln k! - k\ln\alpha + \alpha + \ln k!$$

$$= k\ln\beta/\alpha - \beta - \alpha > t$$

$$k > \frac{t + \alpha + \beta}{\ln\beta/\alpha} = t'$$

$$0.05 = \alpha = P[k > t|H_0] = \sum_{k=t+1}^{\infty} \frac{\alpha^k}{k!}e^{-\alpha} = \sum_{k=t+1}^{\infty} \frac{2^k}{k!}e^{-2}$$

$$\Rightarrow t = 4$$

(b)

$$P_D = P[k > 4|H_1] = \sum_{k=5}^{\infty} \frac{6^k}{k!}e^{-6} = 0.714$$

(c) If we take n measurements the test becomes

$H_0: \alpha = 2n$ $\qquad 0.05 = \sum_{k=t+1}^{\infty} \frac{(2n)^k}{k!}e^{-2n}$

$H_1: \beta = 6n$

Try $n = 2 \Rightarrow t = 7$

$$P_D = [k > 7|H_1] = \sum_{k=8}^{\infty} \frac{12^k}{k!}e^{-6} = 0.91$$

8.54) Assume $\bar{X}_n$ is used in test

H_0: Gaussian $m=8$ $\sigma^2 = 1/n$

H_1: Gaussian $m=9$ $\sigma^2 = 1/n$

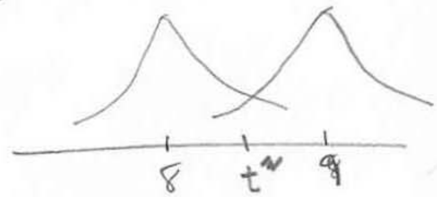
Apply Neyman - Pearson criterion:

$$\ln \Lambda(x) = -\frac{1}{2}\ln 2\pi/n - \frac{1}{2}\frac{(x-9)^2}{1/n} + \frac{1}{2}\ln\frac{2\pi}{n} + \frac{1}{2}\frac{(x-8)^2}{1/n} \underset{H_0}{\overset{H_1}{\gtrless}} t$$

$$-(x-9)^2 + (x-8)^2 \underset{H_0}{\overset{H_1}{\gtrless}} t'$$

$$-x^2 + 18x - 81 + x^2 - 16x + 64 \underset{H_0}{\overset{H_1}{\gtrless}} t'$$

$$x \underset{H_0}{\overset{H_1}{\gtrless}} t''$$



$$\alpha = 0.01 = P[X > t'' \mid H_0]$$

$$= \int_{t''}^{\infty} \frac{1}{\sqrt{2\pi/n}} e^{-(x-8)^2/2/n}\,dx = \underbrace{Q\left(\frac{t''-8}{1/\sqrt{n}}\right)}_{2.326}$$

$$\Rightarrow (t''-8)\sqrt{n} = 2.326$$

$$P_D = 0.99 = P[X > t'' \mid H_1] = \underbrace{Q\left(\frac{t''-9}{1/\sqrt{n}}\right)}_{-2.326}$$

$$\left.\begin{array}{l} \dfrac{t''-8}{\sqrt{n}} = 2.326 \\[2ex] \dfrac{t''-9}{\sqrt{n}} = -2.326 \end{array}\right\} \Rightarrow n = 21.64 \text{ use } n = 22$$

then $t'' = 8 + \frac{2.326}{\sqrt{22}}$

$= 8.4959$

8.55

H_0: exponential $m=2$ $\quad f_X(x) = \lambda e^{-\lambda x}$

H_1: exponential $m=4$

Neyman Pearson:

$$\ln \Lambda(x) = \ln \tfrac{1}{4} - x/4 - \ln \tfrac{1}{2} + x/2 \underset{H_0}{\overset{H_1}{\gtrless}} t$$

$$x \underset{H_0}{\overset{H_1}{\gtrless}} t'$$

(a) $$\alpha = 0.05 = P[X > t' \mid H_0] = \int_{t'}^{\infty} \tfrac{1}{2} e^{-x/2}\,dx = e^{-t'/2}$$

$$\Rightarrow \ln 0.05 = -t'/2$$

$$\Rightarrow t' = -2\ln 0.05 = 5.9915$$

(b) $$P_D = P[X > t' \mid H_1] = \int_{t'}^{\infty} \tfrac{1}{4} e^{-x/4}\,dx = e^{-5.9915/4} = 0.22$$

Difficult to identify heavy users without misidentifying light users.

8.56) H_0: Pareto $m=3$ $a=3$ $\quad x_m = \frac{m(a-1)}{a} = 2$ for both distributions

H_1: Pareto $m=16$ $a=8/7$

$$f_X(x) = a\frac{x_m^a}{x^{a+1}} \quad x \geq 2$$

a) Neyman-Pearson

$$\ln\Lambda(x) = \ln\frac{8}{7} + \frac{8}{7}\ln x_m - \frac{15}{7}\ln x - \ln 3 + 3\ln x_m + 4\ln x \underset{H_0}{\overset{H_1}{\gtrless}} t$$

$$-\frac{15}{7}\ln x + 4\ln x \underset{H_0}{\overset{H_1}{\gtrless}} t'$$

$$\frac{13}{7}\ln x \underset{H_0}{\overset{H_1}{\gtrless}} t''$$

$$x \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

$$\alpha = 0.01 = P[X > \gamma \mid H_0] = \frac{2^3}{\gamma^3}$$

$$\Rightarrow \gamma = \left(\frac{8}{.01}\right)^{1/3} = 9.283$$

b) $$P_D = P[X > \gamma \mid H_1] = \frac{2^{8/7}}{\gamma^{8/7}} = 0.173$$

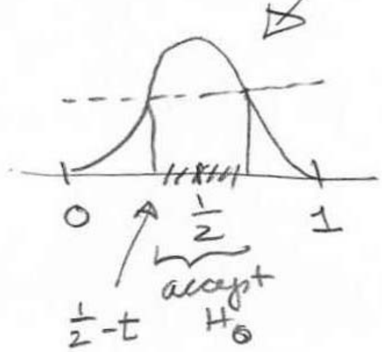
8.57

H_0: Beta $a = b = 10$ $f(x|H_0) = c\,x^9(1-x)^9$ $0<x<1$

H_1: Beta $a = b = 5$ $f(x|H_1) = c'\,x^5(1-x)^5$

$$\Lambda(x) = \frac{f(x|H_1)}{f(x|H_0)} = \frac{c' x^5(1-x)^5}{c\,x^9(1-x)^9} = \frac{c'}{c\,x^4(1-x)^4} \underset{H_0}{\overset{H_1}{\gtrless}} t$$

$$\left|x - \tfrac{1}{2}\right| \underset{H_0}{\overset{H_1}{\gtrless}} t' \quad \Leftarrow$$



$$\alpha = 0.05 = P\left[\left|X - \tfrac{1}{2}\right| > t \,\middle|\, H_0\right]$$

$$= 2\int_0^{\frac{1}{2}-t} c\,x^9(1-x)^9\,dx$$

Use Octave
beta_inv(0.025, 10, 10)

$$\Rightarrow \tfrac{1}{2} - t = 0.28864$$

$$\Rightarrow t = .31136$$

$$P_D = P\left[\left|X - \tfrac{1}{2}\right| > t \,\middle|\, H_1\right] = 2\int_0^{\frac{1}{2}-t} c' x^5(1-x)^5\,dx$$

$$= 0.17098$$

Use octave
beta_cdf(0.28864, 5, 5)

8.58 $m_0 = 2 \quad m_1 = 1.75 \quad \sigma^2 = 0.04 \quad n = 10 \quad \bar{x}_{10} = 1.82$

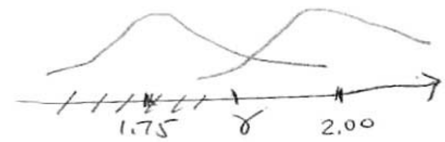
(a) H_0: Gaussian $m_0 = 2$ $\sigma^2 = 0.04$

H_1: Gaussian $m_1 = 1.75$ $\sigma^2 = 0.04$

Simple Binary Hypothesis Test as in Example 8.25

$$\ln \Lambda(x) = (m_1 - m_0)\bar{X}_n + c \underset{H_0}{\overset{H_1}{\gtrless}} t$$

$$\bar{X}_n \underset{H_0}{\overset{H_1}{\lessgtr}} \gamma$$



$$0.05 = \alpha = P[\bar{X}_n < \gamma \mid H_0] = 1 - Q\left(\underbrace{\sqrt{n}\,\frac{\gamma - 2}{\sigma_X}}\right) \quad \text{Reject } H_0$$

$$\frac{\sqrt{10}(\gamma - 2)}{0.2} = -1.644$$

$$\Rightarrow \gamma = 1.896 \qquad \bar{X}_n = 1.82 \Rightarrow \text{Reject } H_0$$

(b) $$P_D = P[\bar{X}_n < \gamma \mid H_1] = \int_{-\infty}^{\gamma} \frac{1}{\sqrt{2\pi}\sigma_X} e^{-(x-1.75)^2/2\sigma_X^2}\,dx = 1 - Q\left(\sqrt{10}\,\frac{\gamma - 1.75}{.2}\right)$$

$$= 0.9895$$

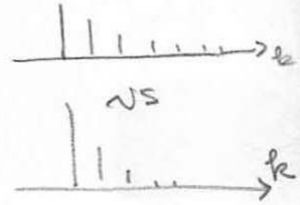
(c) $\bar{X}_n = 1.82 \qquad p = P[\bar{X}_n - 2 < 1.82] = 1 - Q\left(\frac{\sqrt{10}(1.82 - 2)}{0.2}\right)$

$$= 0.00221$$

The p-value is lower than $\alpha = 0.05$ or even $\alpha = 0.01$

8.59 $H_0: p = 1/2$ $\qquad P[X=k \mid p] = p(1-p)^{k-1} \quad k=1,2,\dots$

$H_1: p = 3/4$



(a) $$\Lambda(k) = \frac{\frac{3}{4}\left(\frac{1}{4}\right)^{k-1}}{\left(\frac{1}{2}\right)^{k}} = \frac{3}{4}\left(\frac{4}{1}\right)\left(\frac{1}{2}\right)^{k} \underset{H_0}{\overset{H_1}{\gtrless}} t$$

$$\Leftrightarrow \quad k \ln\frac{1}{2} \underset{H_0}{\overset{H_1}{\gtrless}} t'$$

$$\Leftrightarrow \quad k \underset{H_0}{\overset{H_1}{\lessgtr}} \gamma \qquad \text{since } \ln\frac{1}{2} < 0$$

$\Rightarrow$ Reject if $k < \gamma$

$$0.05 = \alpha = P[X < \gamma \mid H_0] = \sum_{k=1}^{\gamma-1}\left(\frac{1}{2}\right)^{k} = \frac{1}{2} + \frac{1}{4} + \dots + \left(\frac{1}{2}\right)^{\gamma-1}$$

Unless we let the rejection region be $\{k < 1\}$ - all values of k, we cannot find a region that satisfies the above equation.

(b) for $H_0 = p = 1/2$ $\qquad k \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$

$H_1 = p = 1/4$

$$0.05 = \alpha = P[X \geq \gamma \mid H_0] = \sum_{k=\gamma}^{\infty}\left(\frac{1}{2}\right)^{k} = \left(\frac{1}{2}\right)^{\gamma-1}$$

$$\ln 0.05 = (\gamma-1)\ln\frac{1}{2}$$

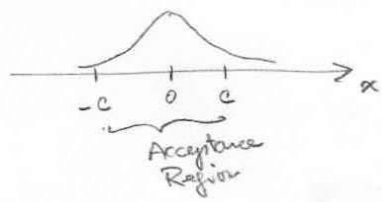
$$\gamma = 1 + \frac{\ln 0.05}{\ln 1/2} = 5.3219 \Rightarrow \gamma = 6$$

$$P_D = P[X \geq \gamma \mid H_0] = \sum_{k=\gamma}^{\infty}\left(\frac{3}{4}\right)^{k-1}\left(\frac{1}{4}\right) = \left(\frac{3}{4}\right)^{\gamma-1} = \left(\frac{3}{4}\right)^{5} = 0.237$$

low

8.60

H_0: Gaussian $m=0$ $\sigma^2 = 1/n$
H_1: Gauss $m \neq 0$ $\sigma^2 = 1/n$



(a) Proceeding as in Ex. 8.28

$$0.10 = \alpha = P[\,|\bar{X}_n| > c \mid H_0] = 2Q(c\sqrt{n})$$

$$c = z_{\alpha/2}/\sqrt{n} = 1.644/\sqrt{n}$$

(b) $P[\text{Type II error}] = P[\,|\bar{X}_n| < c \mid m = \mu \neq 0]$

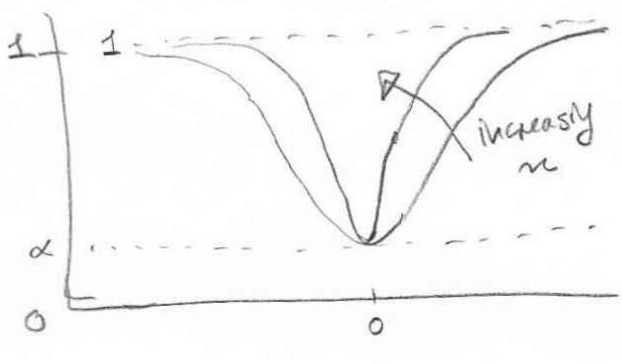
$$= Q(-z_{\alpha/2} - \sqrt{n}\mu) - Q(z_{\alpha/2} - \sqrt{n}\mu)$$
$$= Q(-1.644 - \sqrt{n}\mu) - Q(1.644 - \sqrt{n}\mu)$$
$$= \beta(\mu)$$

Power of test $= 1 - \beta(\mu)$

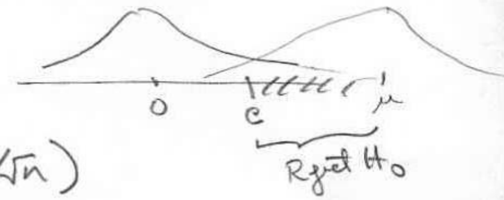
(c) The following Octave commands plot the power curve for n=64

```
> mu = [-10: 0.10 :10]
> plot (mu, 1.-(-normal_cdf (-1.6449.-8mu)
                  +normal_cdf (1.6449.-8mu)))
```

We obtain



8.61 H_0: Gaussian $m=0$ $\sigma^2=1/n$
H_1: Gaussian $m>0$ $\sigma^2=1/n$



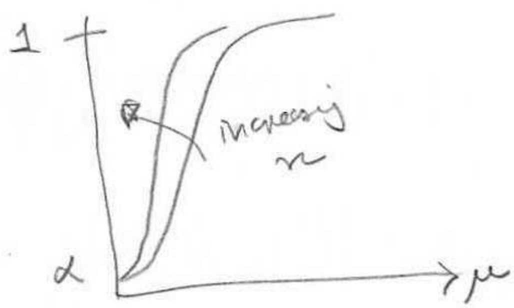
(a) $0.10=\alpha=P[\bar{X}_n > c \mid H_0] = Q(c/\sqrt{n})$

$\Rightarrow c = z_\alpha/\sqrt{n} = 1.2816/\sqrt{n}$

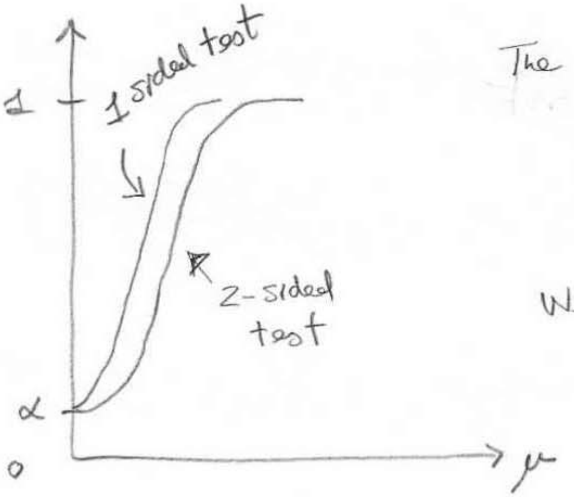
(b) $P[\text{Type II error}] = P[\bar{X}_n < c \mid H_1] = 1 - Q(1.2816 - \sqrt{n}\mu)$

(c) The following commands plot the power curve

```
> mu=[0:0.1:10]
> plot(mu, 1.-normal_cdf(1.2816.-2*mu))
```

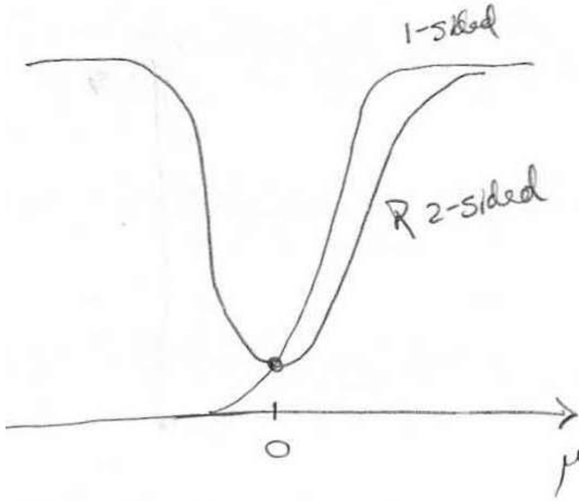


8.62) Using the Octave commands in 8.60 and 8.61 we obtain the following



The 1-sided test is more powerful in the region $\mu > 0$, than the 2-sided test

We already saw that the 1-sided test is UMP for $\mu > 0$



If we compare the power of the 1-sided test for $\mu > 0$ vs the 2-sided test for values $\mu < 0$ we obtain the curve on the left

The 1-sided test is useless for values $\mu < 0$

∴ The 2-sided test strikes a compromise to perform well for all $\mu \neq 0$, and consequently it is unable to outperform the more specialized 1-sided test in the latter's design region.

8.65

$H_0: p = \frac{1}{2}$ $\quad n = 100$ tosses

$H_1: p \neq \frac{1}{2}$ $\quad$ Let $k = \#$ heads.

a (i) $\Lambda(k) = \dfrac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k}\left(\frac{1}{2}\right)^n} \underset{H_0}{\overset{H_1}{\gtrless}} t \iff \left(\dfrac{p}{1-p}\right)^k \underset{H_0}{\overset{H_1}{\gtrless}} t'$

$\iff k \ln \dfrac{p}{1-p} \underset{H_0}{\overset{H_1}{\gtrless}} t' \Rightarrow \begin{cases} k \underset{H_0}{\overset{H_1}{\gtrless}} t'' & p > \frac{1}{2} \\ k \underset{H_0}{\overset{H_1}{\lessgtr}} & p < \frac{1}{2} \end{cases}$

$\iff \left|k - \underbrace{\tfrac{n}{2}}_{50}\right| \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$

$$\alpha = 0.01 = P\left[|k-50| > \gamma \,|\, H_0\right] \approx P\left[\left|\frac{X-50}{\sqrt{25}}\right| > \underbrace{\frac{\gamma}{\sqrt{25}}}_{2.5758}\right]$$

$\Rightarrow \gamma = 2.5758\sqrt{25} = 12.88$

$\Rightarrow$ Use $\gamma = 13$

b(i) $P[\text{Type II error}] = P\left[\,|k-50| < 13 \,|\, H_1\right]$

$$\approx \int_{37}^{63} \frac{1}{\sqrt{2\pi np(1-p)}} e^{-(x-np)/2np(1-p)}\,dx$$

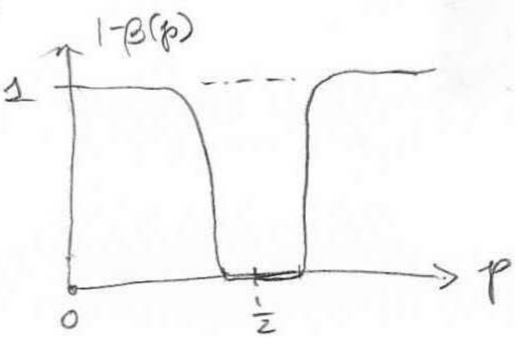
$$\approx Q\left(\frac{37-100p}{10\sqrt{p(1-p)}}\right) - Q\left(\frac{63-100p}{10\sqrt{p(1-p)}}\right)$$

$\approx \beta(p)$

Power $= 1 - \beta(p)$

(ii) & (iii) done similarly

Conclusions of comparison as in Prob. 8.62



8.67 H_0: X Gaussian $m \le \mu$ σ_X^2 known — composite hypotheses
H_1: X Gaussian $m > \mu$ σ_X^2 known

Use the following decision regions

Accept Reject

$$\mu + \frac{z_\alpha \sigma}{\sqrt{n}}$$

$$P[\text{Type I error}] = P\left[X < \mu + \frac{z_\alpha \sigma}{\sqrt{n}} \,\middle|\, H_0\right] \qquad \mu' < \mu$$

$$= \int_{-\infty}^{\mu + \frac{z_\alpha \sigma}{\sqrt{n}}} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu')/2\sigma^2} dx$$

$$= 1 - Q\left(\frac{\mu - \mu' + \frac{z_\alpha \sigma}{\sqrt{n}}}{\sigma/\sqrt{n}}\right)$$

$$= 1 - Q\Big(\underbrace{\frac{\mu - \mu'}{\sigma/\sqrt{n}}}_{>0} + z_\alpha\Big)$$

$$\le 1 - Q(z_\alpha) = \alpha \quad ✓$$

8.68 $m = 2 \quad n=10 \quad \bar{X}_{10} = 2.2 \quad \hat{\sigma}^2_{10} = 0.04$

Proceeding as in Ex. 8.29

$H_0: m=2$ σ^2 unknown
$H_1: m \neq 2$ σ^2 unknown

$$T = \frac{\bar{X}_n - m}{\hat{\sigma}_n/\sqrt{n}} = \sqrt{10}\,\frac{\bar{X}_n - 2}{(.2)}$$

@ $\alpha = 0.05$, degree $10-1=9$ $t_{0.025,9} = 2.2622$

$\therefore$ Accept H_0 if $\left|\frac{x-2}{\hat{\sigma}_n/\sqrt{n}}\right| \leq 2.2622$

In the example we have $T_0 = \sqrt{10}\,\frac{2.2-2}{.2} = 3.1623 > 2.2622$

$\Rightarrow$ Reject H_0

$$p = P\left[\,\left|\frac{x-2}{\hat{\sigma}_n/\sqrt{n}}\right| > 3.1623\right] = 2F_9(3.1623)$$
$$= 0.011508$$

which is lower than $\alpha = 0.05$

8.69

H_0: mean 50 variance unknown $n=8$

H_1: mean 55 variance unknown $\bar{X}_8 = 52.5$

We assume that X has a Gaussian distribution $\hat{\sigma}_8 = 3$
and use the Student-t statistic

$$T = \frac{\bar{X} - m}{\sigma/\sqrt{n}}$$

Accept H_0 if $\bar{X} < \gamma$

$$\alpha = P[\bar{X} > \gamma \mid H_0] = P\left[\frac{\bar{X}-50}{\hat{\sigma}_n/\sqrt{n}} > \frac{\gamma - 50}{\hat{\sigma}_n/\sqrt{n}}\right]$$

$$= 1 - F\underbrace{\left(\frac{\gamma-50}{\hat{\sigma}_n/\sqrt{n}}\right)}_{t_{\alpha, n-1}}$$

$\alpha = 0.01 \Rightarrow t_{0.01,7} = 2.998$

$\alpha = 0.05 \Rightarrow t_{0.05,7} = 1.8946$

$$\alpha = 0.01 \quad \gamma = 50 + \frac{\sqrt{n}\, t_{\alpha,n-1}}{\hat{\sigma}_n} = 50 + \frac{\sqrt{8}\,(2.998)}{3} = 52.824$$

$\Rightarrow$ Accept $\bar{X}_8 = 52.5$

$$\alpha = 0.05 \quad \gamma = 50 + \frac{\sqrt{8}(1.8946)}{3} = 51.7862$$

$\Rightarrow$ Reject $\bar{X}_8 = 52.5$

(b) $$p = P\left[\frac{\bar{X}-50}{\hat{\sigma}_n/\sqrt{n}} > \frac{52.5-50}{3/\sqrt{8}}\right] = 0.0252$$

less than 0.05
but greater than 0.01

8.70) $H_0: m = 4$ $\quad \bar{X}_n = 3.3 \quad n = 100$

$H_1: m < 4 \quad \hat{\sigma}_n = \frac{1}{2}$

(a) Assume $\bar{X}_n$ Gaussian since n is large

This is a one-sided test:

Accept H_0 if $\bar{X}_n > \gamma$

Reject H_0 if $\bar{X}_n < \gamma$

$$\gamma = 4 - \frac{\sigma}{\sqrt{n}} z_\alpha = 4 - \frac{1}{2(10)} z_\alpha, \quad z_{0.01} = 2.3263, \quad z_{0.05} = 1.6449$$

$$\gamma = \begin{cases} 3.8837 & \alpha = 0.01 \\ 3.9178 & \alpha = 0.05 \end{cases}$$

Both tests reject H_0 for $\bar{X} = 3.3$

Frosh rule!

(b) $$p = P[\bar{X}_n < 3.3 \mid H_0] = Q\left(\frac{3.3 - 4}{(\frac{1}{2})/10}\right) = Q(.7(20))$$

$$= Q(14) \simeq e^{-(14)^2/2} = 0$$

8.75

H_0: Gaussian $m=0$ $\sigma^2=4$

H_1: Gaussian $m=0$ $\sigma^2<4$

(a) Accept H_0 if $\hat{\sigma}_n^2 > \gamma$

Reject H_0 if $\hat{\sigma}_n^2 < \gamma$

$$\alpha = P[\hat{\sigma}_n^2 < \gamma \mid H_0] = P\left[\frac{(n-1)\hat{\sigma}_n^2}{\sigma_0^2} < \frac{(n-1)\gamma}{\sigma_0^2}\right] = 1-\chi^2$$

$$= 1 - P\left[\chi^2 > \frac{(n-1)\gamma}{\sigma_0^2}\right]$$

$$1-\alpha = P\left[\chi^2 > \underbrace{\frac{(n-1)\gamma}{\sigma_0^2}}_{\chi^2_{1-\alpha,\,n-1}}\right]$$

$$\chi^2_{1-\alpha,\,n-1} = \chi^2_{.99,\,n-1} = 1.2390$$

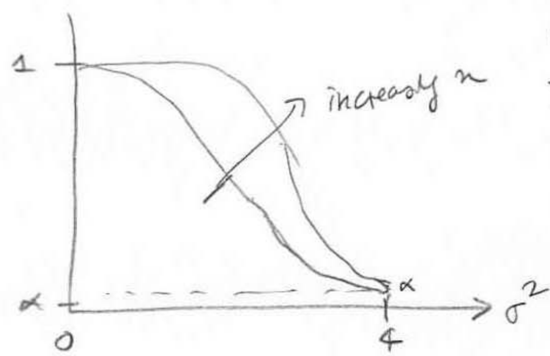
$$\gamma = \frac{\chi^2_{.99,\,n-1}}{n-1}\sigma_0^2$$

n	8	64	256
$\chi^2_{.99,\,n-1}$	1.2390	39.85	205.4
γ	0.708	2.53	3.22

(b) Power $= P[\hat{\sigma}_n^2 < \gamma \mid H_1]$

$$= P\left[\frac{(n-1)\sigma_n^2}{\sigma^2} < \frac{(n-1)\gamma}{\sigma^2}\right] = P\left[\chi^2 < \frac{(n-1)\gamma}{\sigma^2}\right]$$

for σ^2 not σ_0^2



octave:

```
> sig2 = [0:0.1:4];
> plot(sig2, chisquare_cdf(63*2.53./sig2, 63));
```

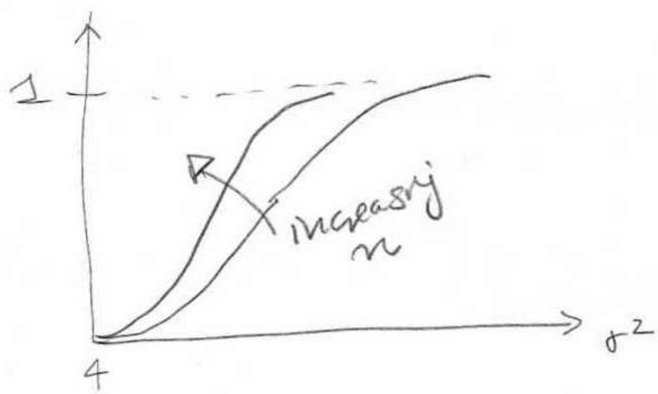
8.76 H_0: Gaussian $m=0$ $\sigma^2=4$
H_1: Gauss $m=0$ $\sigma^2>4$

a) Accept H_0 if $\sigma_n^2<\gamma$

$$\alpha = P[\sigma_n^2>\gamma \mid H_0] = P[\chi^2 > \underbrace{\frac{(n-1)\gamma}{\sigma_0^2}}_{\chi^2_{\alpha,n-1}}]$$

n	8	64	256
$\chi^2_{.01,n-1}$	18.47	92.01	310.46
γ	10.56	5.84	4.87

b) $$\text{Power} = P[\hat{\sigma}_n^2>\gamma \mid H_1] = P[\chi^2 > \frac{(n-1)\gamma}{\sigma^2}]$$
$\uparrow$ for σ^2



> sig2 [4: 0.1: 8]
> plot (sig2, 1 - chisquare_cdf (63 * 5.84. / sig2, 63))

8.77

$H_0: \sigma^2 = \mu_0 = 7$

$H_1: \sigma^2 = \mu \neq \mu_0$

(a) Accept H_0 if $a < \hat{\sigma}_n^2 < b$

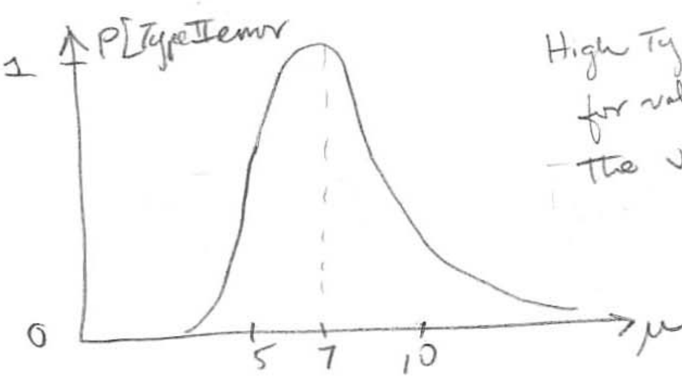
$$\Leftrightarrow \frac{(n-1)a}{\mu_0} < \frac{(n-1)\hat{\sigma}_n^2}{\mu_0} < \frac{(n-1)b}{\mu_0}$$

$$\alpha = 1 - P[a < \hat{\sigma}_n^2 < b \mid H_0] = 1 - P\Big[\underbrace{\underbrace{\frac{(n-1)a}{7}}_{\chi^2_{1-\alpha/2,\, n-1}}}_{\chi^2_{.995,70}} < \chi^2 < \underbrace{\underbrace{\frac{(n-1)b}{7}}_{\chi^2_{\alpha/2,\, n-1}}}_{\chi^2_{.005,70}}\Big]$$

$$a = \frac{\mu_0 \chi^2_{.995, n-1}}{(n-1)} = 4.3275 \qquad b = \frac{\mu_0 \chi^2_{.005, n-1}}{n-1} = 10.42$$

(b) $P[\text{Type II error}] = P[a < \hat{\sigma}_n^2 < b \mid H_1]$

$$= P\left[\frac{70(4.3275)}{\mu} < \frac{(n-1)\sigma_n^2}{\mu} < \frac{70(10.42)}{\mu}\right]$$



High Type II error rates for values of μ in the vicinity of 7

8.6 Bayesian Decision Methods

8.81

H_0: Exponential $m = \frac{1}{2}$ $\quad p_0 = \frac{1}{10}$

H_1: Exponential $m = 5$ $\quad 1 - p_0 = \frac{9}{10}$

$C_{00} = 0 \qquad C_{10} = 3 \qquad\qquad C_{01} - C_{00} = 5$

$C_{01} = 5 \qquad C_{11} = 0 \qquad\qquad C_{10} - C_{11} = 3$

cost of short life sold as long

$$\text{Accept } H_0 \qquad \underbrace{\frac{f(x|H_1)}{f(x|H_0)}} < \frac{\frac{1}{10} \cdot 5}{\frac{9}{10} \cdot 3} = \frac{5}{27}$$

cost of long life sold as short

$$\frac{\frac{1}{5} e^{-x/5}}{2 e^{-2x}} < \frac{5}{27}$$

$$-\frac{x}{5} + 2x < \ln \frac{50}{27}$$

$$\frac{9}{5} x < \ln \frac{50}{27}$$

$$x < \frac{5}{9} \ln \frac{50}{27} = 0.3423$$

8.82 (a) Maximum Likelihood Decision Rule

$x=0$ $p(0|H_1) < p(0|H_0) \Rightarrow$ decide H_0

$x=1$ $p(1|H_1) > p(1|H_0) \Rightarrow$ decide H_1

$x=e$ $p(0|H_1) = p(1|H_0) \Rightarrow$ no clear decision

The cost does not enter in the ML decision rule

We can use knowledge of cost to break the tie when $x=e$

If $C_{10} > C_{01}$ then cost of mistaking 1 for 0 is higher so decide H_1 when $x=e$

This gives

if $C_{10} > C_{01}$

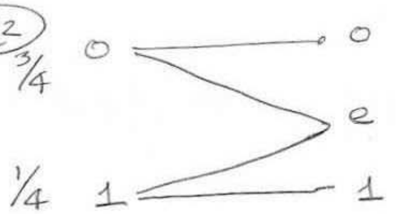
Similarly

if $C_{10} < C_{01}$

These are the rules obtained in the Bayes' case

If $C_{10} = C_{01}$ then we have no basis for deciding one way or another

8.82



$P[0|H_1] = \frac{1}{2}$

$P[e|H_1] = \frac{1}{2}$ $\quad P[e|H_0] = \frac{1}{2}$

$P[1|H_1] = \frac{1}{2}$

H_0: $\Theta = 0$

H_1: $\Theta = 1$

Accept H_0 if $\dfrac{p(x|H_1)}{p(x|H_0)} < \dfrac{P_0\, b\, C_{10}}{P_1\, C_{10}} = 3b$

(b)

Bayes' Decision Rule:

$$\frac{p(0|H_1)}{p(0|H_0)} = \frac{0}{1/2} = 0 \Rightarrow \text{decide } H_0$$

$$\frac{p(1|H_1)}{p(1|H_0)} = \frac{1/2}{0} = \infty \Rightarrow \text{decide } H_1$$

$$\frac{p(e|H_1)}{p(e|H_0)} = 1 \overset{?}{<} 3b$$

$\Rightarrow$ No $b = 1/6$ $\Rightarrow$ decide H_1

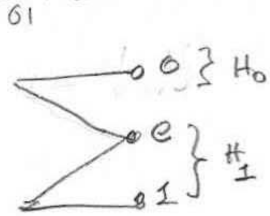
$\Rightarrow$ Yes $b = 6$ $\Rightarrow$ decide H_0

$\Rightarrow$ Yes $b = 1$ decide H_0

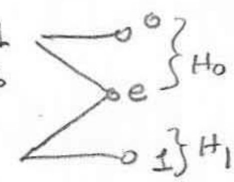
Average cost is:

$$C = C_{01} P[\text{decide } H_1 | H_0] P_0 + C_{10} P[\text{decide } H_0 | H_1] P_1$$

$b = 1/6$



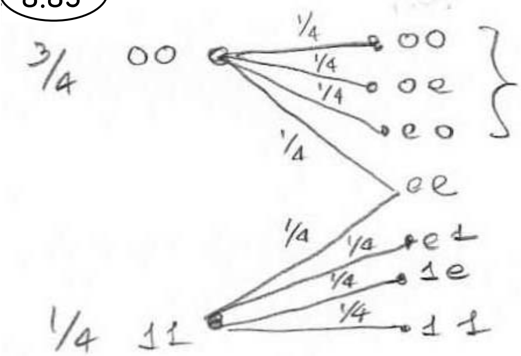
$b = 1$
$b = 6$



$$C = C_{01} \frac{1}{2} \frac{3}{4} = \frac{1}{6} C_{10} \frac{3}{8} = \frac{1}{16} C_{10}$$

$$C = C_{10} \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} C_{10}$$

8.83



$p(\underline{x}|H_0) > 0$ and $p(\underline{x}|H_1) = 0$ $\Rightarrow$ decide H_0

$p(\underline{x}|H_1) > 0$ and $p(\underline{x}|H_0) = 0$ $\Rightarrow$ decide H_1

$$1 = \frac{p(ee|H_1)}{p(ee|H_0)} \overset{?}{<} \frac{p_0 b C_{10}}{p_1 C_{10}} = 3b = \begin{cases} 1/2 & b = 1/6 \\ 3 & b = 1 \\ 18 & b = 6 \end{cases}$$

for $b = \frac{1}{6}$ $\quad \underline{x} = ee \Rightarrow$ decide H_1

$b = 1, 6$ $\quad \underline{x} = ee \Rightarrow$ decide H_0

Average cost:

$b = \frac{1}{6}$ $\quad C = C_{01} \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{1}{6}\frac{3}{16}C_{10} = \frac{1}{32}C_{10}$

$b = 1, 6$ $\quad C = C_{10}\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{C_{10}}{16}$

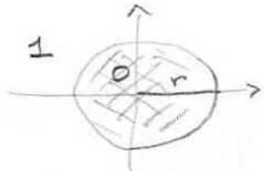
ML Rules give same decision rules as above if cost used to break ties (in this case only)

8.84

Bob H_0: 2D Gaussian mean zero, variance 1 $\quad P_0 = 1/2$ assume
Rick H_1: 2D Gaussian mean zero, variance 4 $\quad P_1 = 1/2$ equal turns

$c_{01} = 1$

$c_{10} = 1$



$$C = c_{01} \cdot P[1|0]\tfrac{1}{2} + c_{10} P[0|1]\tfrac{1}{2}$$

$$= P[1|0]\tfrac{1}{2} + P[0|1]\tfrac{1}{2}$$

Min Cost Rule:

Accept H_0 if $\underbrace{\dfrac{f(\underline{x}|H_1)}{f(\underline{x}|H_0)}}_{\frac{2\pi(1)\, e^{-(x^2+y^2)/8}}{2\pi(8)\, e^{-(x^2+y^2)/2}}} < \dfrac{P_0 c_{01}}{P_1 c_{10}} = 1$

$$e^{\frac{3}{8}(x^2+y^2)} < 8$$

$$x^2 + y^2 < \frac{8}{3}\ln 8 = 5.5452 = R^2$$

$$\Rightarrow R = 2.3548$$

If $c_{01} = 2$, $c_{10} = 1$

$$x^2+y^2 \overset{H_0}{<} \frac{8}{3}\ln 16 = 7.3936$$

$$\Rightarrow R = 2.7191$$

Bob expands his radius to reduce cost.

8.85 H_0: Binomial n, $p = 10^{-3}$ $\quad P[H_0] = 1-\alpha = 4/5$

H_1: Binomial n, $1-p = 1-10^{-3}$ $\quad P[H_1] = \alpha = 1/5$

(a) ML Rule

$$\frac{P[N=k \mid H_1]}{P[N=k \mid H_0]} = \frac{\binom{n}{k}(1-p)^k p^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \left(\frac{1-p}{p}\right)^{2k}\left(\frac{p}{1-p}\right)^n$$

$$= \left(\frac{p}{1-p}\right)^{n-2k} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

$$(n-2k)\underbrace{\ln\underbrace{\left(\frac{p}{1-p}\right)}_{<1}}_{<0} \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$n \underset{H_0}{\overset{H_1}{\lessgtr}} 2k \qquad \boxed{\frac{n}{2} \underset{H_0}{\overset{H_1}{\lessgtr}} k}$$

Majority Rule

(b)

$$\frac{P[N=k \mid H_1]}{P[N=k \mid H_0]} = \left(\frac{1-p}{p}\right)^{2k}\left(\frac{p}{1-p}\right)^n \gtrless \frac{(1-\alpha)\cdot 1}{\alpha \cdot 1}$$

$$(n-2k)\ln\frac{p}{1-p} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{1-\alpha}{\alpha}\right)$$

$$n-2k \underset{H_0}{\overset{H_1}{\lessgtr}} \frac{\ln(1-\alpha)/\alpha}{\ln p/(1-p)} = \frac{1.3863}{-6.91} = -.2006$$

$$\frac{1}{2}(8.2) \underset{H_1}{\overset{H_0}{\lessgtr}} k \qquad \{0,1,\ldots,4\} \Rightarrow H_0$$

$$\{5,6,7,8\} \Rightarrow H_1$$

(c)

$$P[\text{Type I error}] = \sum_{k=5}^{8}\binom{8}{k}(10^{-3})^k(1-10^{-3})^{8-k} \approx \binom{8}{5}10^{-15}$$

$$P[\text{Type II error}] = \sum_{k=0}^{4}\binom{8}{k}(1-10^{-3})(10^{-3})^{8-k} \approx \binom{8}{4}(10^{-12})$$

$$P_e = \binom{8}{5}10^{-15}\cdot\frac{4}{5} + \binom{8}{4}10^{-12}\,\frac{1}{5} \approx \binom{8}{4}10^{-12}\,\frac{1}{5}$$

8.86

H_0: Gaussian $m=-1$ σ^2/n $1-\alpha$

H_1: Gauss $m=+1$ σ^2/n α

a) ML Decision rule

$$\frac{f_X(x|H_1)}{f_X(x|H_0)} = \frac{\sqrt{2\pi\sigma^2/n}\; e^{-(x-1)^2/2\sigma^2/n}}{\sqrt{2\pi\sigma^2/n}\; e^{-(x+1)^2/2\sigma^2/n}} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

take ln of both sides

$$-(x-1)^2 + (x+1)^2 \underset{H_0}{\overset{H_1}{\gtrless}} 0 \iff \boxed{x \underset{H_0}{\overset{H_1}{\gtrless}} 0}$$

Decide based on sign of x

b) Bayes' Decision Rule

$$-\frac{(x-1)^2}{2\sigma^2/n} + \frac{(x+1)^2}{2\sigma^2/n} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{1-\alpha}{\alpha}\right)$$

$$4x \underset{H_0}{\overset{H_1}{\gtrless}} \frac{2\sigma^2}{n}\ln\left(\frac{1-\alpha}{\alpha}\right)$$

$$x \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{2n}\ln\left(\frac{1-\alpha}{\alpha}\right) \triangleq \gamma$$

$$P[\text{Type I}] = P[X>\gamma \mid H_0] = \int_\gamma^\infty \frac{e^{-(x+1)^2/2\sigma^2/n}}{\sqrt{2\pi\sigma^2/n}}dx = Q\left(\frac{\gamma+1}{\sigma/\sqrt{n}}\right)$$

$$P[\text{Type II}] = P[X<\gamma \mid H_1] = \int_{-\infty}^\gamma \frac{e^{-(x-1)^2/2\sigma^2/n}}{\sqrt{2\pi\sigma^2/n}}dx = Q\left(\frac{1-\gamma}{\sigma/\sqrt{n}}\right)$$

$$P_e = Q\left(\frac{\gamma+1}{\sigma/\sqrt{n}}\right)(1-\alpha) + Q\left(\frac{1-\gamma}{\sigma/\sqrt{n}}\right)\alpha$$

for ML Rule we have $\gamma=0$: $P_e^{ML} = Q\left(\frac{\sqrt{n}}{\sigma}\right)$

c) $P[N > 1] = Q\left(\frac{1}{\sigma}\right) = 10^{-3}$

$$\Rightarrow \frac{1}{\sigma} = 3.090$$

Consider the ML Rule:

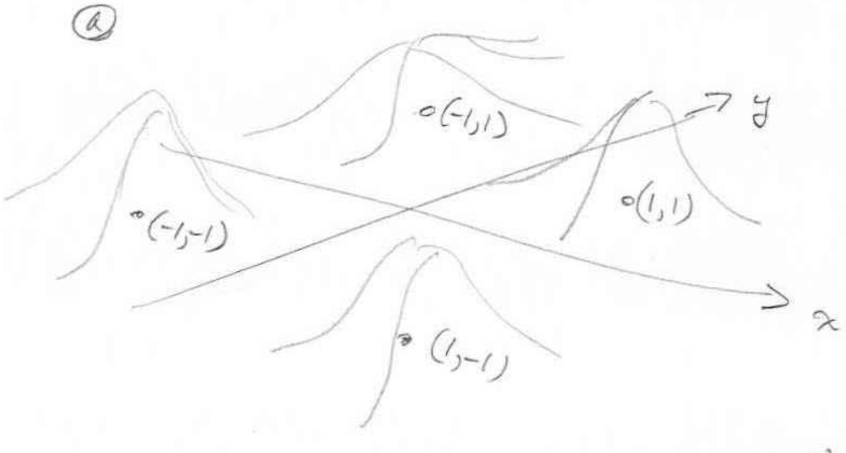
$$P_e = Q\Big(\underbrace{\frac{\sqrt{n}}{\sigma}}_{5.9978}\Big) = 10^{-9}$$

$$\Rightarrow \frac{\sqrt{n}}{3.09} = 5.9978$$

$$n = \left(\frac{5.9978}{3.09}\right)^2 = 3.7676 \Rightarrow \text{Use } \underline{\underline{n=4}}$$

8.87 $f(x,y\,|\,\theta_1,\theta_2) = \frac{1}{2\pi\sigma^2} e^{-[(x-\theta_1)^2 + (y-\theta_2)^2]/2\sigma^2}$

(a)



(b) $C_{ij} = \begin{cases} 1 & i \neq j \\ 0 & i \neq j \end{cases}$ $\quad i = \{1,2,3,4\}$

Quadrant
1 ~ (1,1)
2 ~ (-1,1)
3 ~ (-1,-1)
4 ~ (1,-1)

$$C = \sum_{i=1}^{4}\sum_{j=1}^{4} C_{ij}\, P[\text{decide } j\,|\,H_i]\,P[H_i]$$

$$= \sum_{\substack{i=1 \\ i\neq j}}^{4}\sum_{j=1}^{4} P[\text{Decide } j\,|\,H_i]\,\underbrace{P[H_i]}_{p_i}$$

$$P[\text{Decide } j\,|\,H_i] = \int_{R_j} f(\underline{x}\,|\,H_i)\,d\underline{x} \qquad R_j = \text{Region where we decide } j$$

$$C = \sum_{j=1}^{4}\sum_{\substack{i=1 \\ i\neq j}}^{4} \int_{R_j} f(\underline{x}\,|\,H_i)\,P[H_i]\,d\underline{x}$$

$$= \sum_{j=1}^{4} \underbrace{\int_{R_j}\left(\sum_{\substack{i=1 \\ i\neq j}}^{4} f(\underline{x}\,|\,H_i)\,P[H_i]\right) d\underline{x}}_{\text{contribution to cost (prob of error) by } H_i \text{ not selected in region } R_j}$$

(8.87) – continued –

C is minimized by selecting for each $\underline{x}$ the index j that maximizes

$$P(H_i \mid \underline{x}) = \frac{f(\underline{x} \mid H_i) P[H_i]}{f(\underline{x})}$$

(d) If $P[H_i] = 1/4$ all i, then

$$P[H_i \mid \underline{x}] = \frac{f(\underline{x} \mid H_i)\, 1/4}{f(x)}$$

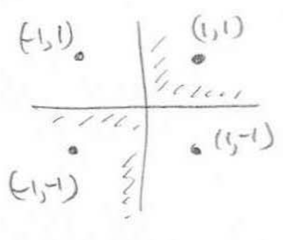
so maximizing $P[H_i \mid \underline{x}]$ is same as maximizing $\underbrace{f(\underline{x} \mid H_i)}_{\text{max likelihood}}$

Compare 2 pdf's

1st quad $f(\underline{x} \mid 11) = \frac{1}{2\pi\sigma^2} e^{-((x-1)^2 + (y-1)^2)/2\sigma^2}$

4th quad $f(\underline{x} \mid 1\,{-1}) = \frac{1}{2\pi\sigma^2} e^{-((x-1)^2 + (y+1)^2)/2\sigma^2}$

For a point (x, y) $f(\underline{x} \mid 11)$ is larger than $f(\underline{x} \mid 1\,{-1})$ if its exponent former's exponent is smaller than the latter's, that is, if



$$\underbrace{(x-1)^2 + (y-1)^2}_{\substack{\text{distance from} \\ (1,1) \text{ to } (x,y)}} < \underbrace{(x-1)^2 + (y+1)^2}_{\substack{\text{distance from} \\ (1,-1) \text{ to } (x,y)}}$$

$\therefore$ the ML decision rule picks the pt $(\pm 1, \pm 1)$ closest to (x, y). The resulting 4 decision regions are the 4 quadrants associated with the 4 signal points.

8.88 $C(g(\underline{X}),\Theta) = |g(\underline{X}) - \Theta|$ ← estimator

$$E[C(g(\underline{X}),\Theta)] = \int_{\underline{x}}\int_{T} |\theta - g(\underline{x})|\, f_\theta(\theta|\underline{x}) f_X(\underline{x})\, d\theta\, d\underline{x}$$

$$= \int_{\underline{x}} \underbrace{\left[\int_{-\infty}^{g(\underline{x})} (g(\underline{x})-\theta) f_\Theta(\theta|\underline{x})\,d\theta + \int_{g(\underline{x})}^{\infty} (\theta - g(\underline{x})) f_\theta(\theta|\underline{x})\,d\theta\right]}_{\text{need to minimize this expression for each } \underline{x}} f_X(\underline{x})\,d\underline{x}$$

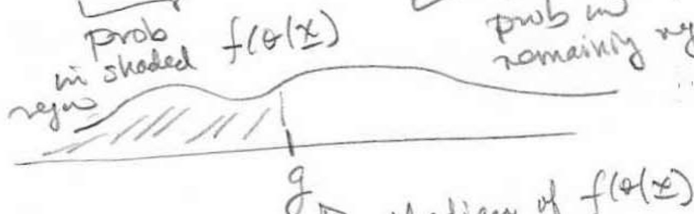
For each $\underline{x}$ we need to minimize

$$\int_{-\infty}^{g} (g-\theta) f(\theta|\underline{x})\,d\theta + \int_{g}^{\infty} (\theta - g) f(\theta|\underline{x})\,d\theta$$

Take derivative with respect to g

$$0 = \int_{-\infty}^{g} f(\theta|\underline{x})\,d\theta - \int_{g}^{\infty} f(\theta|\underline{x})\,d\theta$$

$$\Rightarrow \underbrace{\int_{-\infty}^{g} f(\theta|\underline{x})\,d\theta}_{\text{prob in shaded region}} = \underbrace{\int_{g}^{\infty} f(\theta|\underline{x})\,d\theta}_{\text{prob in remaining region}}$$



$\therefore$ $g(\underline{x})$ is median of $f(\theta|\underline{x})$.

8.89

$$C(g(\underline{x}),\Theta)=\begin{cases}1 & \text{if } |g(\underline{x})-\theta|>\delta\\ 0 & \text{if } |g(x)-\theta|<\delta\end{cases}$$

$$E[C(g(\underline{x}),\Theta)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_{\{|g(\underline{x})-\theta|>\delta\}}\, f(\theta|\underline{x})f(\underline{x})\,d\theta\,d\underline{x}$$

$$= \underbrace{\int_{-\infty}^{\infty}\left[1-\underbrace{\int_{g(\underline{x})-\delta}^{g(\underline{x})+\delta} f(\theta|\underline{x})\,d\theta}_{\text{maximize this}}\right]f(\underline{x})\,d\underline{x}}_{\text{then minimize this}}$$

$\Rightarrow$ $g(\underline{x})$ selects θ so that $f(\theta|\underline{x})$ is max

This is the maximum a posteriori estimate

8.90 $X_1 \ldots X_n$ iid $E[X] = \Theta$ $\sigma_X^2 = 1$

Θ Gaussian mean 0 and unit variance

$$f_{\underline{X}}(\underline{x}|\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(x_i-\theta)^2/2} = \frac{1}{\sqrt{2\pi}^n} e^{-\sum_{i=1}^{n}(x_i-\theta)^2/2}$$

$$f(\underline{x},\theta) = \frac{1}{\sqrt{2\pi}^n} e^{-\sum_{i=1}^{n}(x_i-\theta)^2/2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\theta^2/2}$$

$$= c\, e^{-\left[\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2 + \frac{1}{2}\theta^2\right]}$$

$$= c\, e^{-\frac{1}{2}\left[\sum_{i=1}^{n}(x_i^2 - 2\theta x_i + \theta^2) + \theta^2\right]}$$

$$= c\, e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2} e^{-\frac{1}{2}\left[2\theta^2 - 2\theta\sum_{i=1}^{n} x_i\right]}$$

$$= c\, e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2} e^{-\left[\theta - \frac{1}{2}\sum_{i=1}^{n} x_i\right]^2} e^{\frac{1}{4}\left(\sum_{i=1}^{n} x_i\right)^2}$$

$$= c'\, e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2} e^{\frac{1}{4}\left(\sum_i x_i\right)^2} \frac{e^{-\left[\theta - \frac{1}{2}\sum_{i=1}^{n} x_i\right]^2/2\cdot\frac{1}{2}}}{\sqrt{2\pi\left(\frac{1}{2}\right)}}$$

$$f(\underline{x}) = c'\, e^{-\frac{1}{2}\left[\sum_{i=1}^{n} x_i^2 - \frac{1}{2}\left(\sum_i x_i\right)^2\right]} \underbrace{\int_{-\infty}^{\infty} \frac{e^{-\left(\theta - \frac{1}{2}\sum_i\right)^2/2\cdot\frac{1}{2}}}{\sqrt{2\pi\left(\frac{1}{2}\right)}} d\theta}_{1}$$

$$\Rightarrow f(\theta|\underline{x}) = \frac{c\, e^{-\frac{1}{2}\left[\left(\sum_i x_i^2 - 2\theta x_i + \theta^2\right) + \theta^2\right]}}{c'\, e^{-\frac{1}{2}\left[\sum_i x_i^2 - \frac{1}{2}\left(\sum_i x_i\right)^2\right]}}$$

$$= \frac{c}{c'}\, e^{-\frac{1}{2}\left[-\frac{1}{2}\left(\sum_i x_i\right)^2 - 2\theta\sum_i x_i + (n+1)\theta^2\right]}$$

8.90 – continued –

$$f(\theta|\underline{x}) = \frac{c}{c'} e^{-\frac{1}{2}(n+1)\left[\theta^2 - 2\frac{1}{n+1}\theta\sum x_i + \frac{1}{n+1}\left(\sum x_i^2\right)\right]}$$

$$= \frac{c}{c'} c''(\underline{x}) \; e^{-\frac{1}{2}(n+1)\left[\theta - \frac{1}{n+1}\sum x_i\right]^2}$$

$$E[\theta|\underline{x}] = \frac{1}{n+1}\sum_{i=1}^{n} x_i = \frac{1}{1+\frac{1}{n}}\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)}_{\bar{X}_n}$$

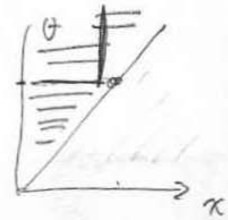
for Gaussian median = mean

∴ the minimum absolute error estimator $= E[\theta|\underline{x}]$

The maximum of the Gaussian occurs at the mean

∴ MAP estimator is also $E[\theta|\underline{x}]$

8.91 X uniform in $[0, \Theta]$. $\quad f_\Theta(\theta) = \theta e^{-\theta}$



(a)
$$f(x,\theta) = f(x|\theta)\, f(\theta) = \frac{1}{\theta}\,\theta e^{-\theta} = e^{-\theta} \qquad 0<x<\theta,\ \theta>0$$

$$f(x) = \int_x^\infty e^{-\theta}\, d\theta = e^{-x}$$

$$f(\theta|x) = \frac{f(x,\theta)}{f(x)} \qquad 0<x<\theta$$

$$= \frac{e^{-\theta}}{e^{-x}} = e^{-(\theta-x)} \qquad \theta \geq x \geq 0$$

$$\frac{1}{2} = \int_x^{g(x)} e^{-(\theta-x)}\, d\theta = \int_0^{g(x)-x} e^{-y}\, dy = 1 - e^{-g(x)+x}$$

$$\frac{1}{2} = e^{-g(x)+x} \qquad -g(x)+x = \ln\frac{1}{2}$$

$$\Longrightarrow \boxed{g(x) = x - \ln\frac{1}{2} = x + \ln 2}$$

$$E[\theta|X] = \int_x^\infty \theta e^{-(\theta-x)}\, dx$$

$$= \int_0^\infty (y+x)\, e^{-y}\, dy = x + 1$$

8.92 X Binomial n, Θ $\quad \Theta \sim$ Beta α, β

a) $$P[k, \Theta] = \binom{n}{k} \theta^k (1-\theta)^{n-k} \, c\, \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

$$P[k] = c\binom{n}{k} \underbrace{\int_0^1 \theta^{k+\alpha-1} (1-\theta)^{n-k+\beta-1} \, d\theta}_{B(k+\alpha-1,\, n-k+\beta-1)}$$

$$f(\Theta|k) = \frac{c\binom{n}{k} \theta^{k+\alpha-1}(1-\theta)^{n-k+\beta-1}}{c\binom{n}{k} B(k+\alpha-1,\, n-k+\beta-1)}$$

$$= \frac{\theta^{k+\alpha-1}(1-\theta)^{n-k+\beta-1}}{B(k+\alpha-1,\, n-k+\beta-1)} \quad \text{Beta pdf.}$$

$$E[\Theta|k] = \frac{k+\alpha}{k+\alpha+n-k+\beta} = \frac{k+\alpha}{\alpha+n+\beta} \quad \checkmark$$

8.93

$P[k|\theta] = \binom{n}{k} \theta^k (1-\theta)^{n-k} \qquad f_\theta(\theta) = 1 \quad 0 < \theta < 1$

$$C(g(x), \theta) = \frac{(\theta - g(x))^2}{\theta(1-\theta)}$$

$$E[C(g(x), \theta)] = \int_0^1 \sum_{k=0}^{n} \frac{(\theta - g(k))^2}{\theta(1-\theta)} \binom{n}{k} \theta^k (1-\theta)^{n-k} \, d\theta$$

$$= \sum_{k=0}^{n} \int_0^1 \underbrace{(\theta - g(k))^2}_{\text{Sqd Error}} \underbrace{\binom{n}{k} \theta^{k-1} (1-\theta)^{n-k-1}}_{\text{Beta Dist.}} \, d\theta$$

minimized by letting

$$g(k) = E[\theta | k] = \frac{k}{k + n - k} = \frac{k}{n}$$

8.7 Testing the Fit of a Distribution to Data

8.94

	Obs.	Expected	$(0-\varepsilon)^2/\varepsilon$
0	0	10.5	10.5
1	0	10.5	10.5
2	24	10.5	17.36
3	2	10.5	6.88
4	25	10.5	20.02
5	3	10.5	5.36
6	32	10.5	44.02
7	15	10.5	1.93
8	2	10.5	6.88
9	2	10.5	6.88
	105		$D^2 = 130.33$

\# degrees of freedom = 9
1% significance level $\Rightarrow$ 21.7
$D^2 > 21.7$
$\Rightarrow$ Reject hypothesis that #'s are unif. dist. in $\{0, 1, \ldots, 9\}$

	Obs.	Expected	$(0-\varepsilon)^2/\varepsilon$
2	24	105/8	9.01
3	2	105/8	9.43
4	25	105/8	10.74
5	3	105/8	7.81
6	32	105/8	77.41
7	15	105/8	0.27
8	2	105/8	9.43
9	2	105/8	9.93
	105		83.26

\# degrees of freedom = 9
1% significance level $\Rightarrow$ 21.7
$D^2 > 21.7$
$\Rightarrow$ Reject hypothesis that #'s are unif. dist. in $\{0, 1, \ldots, 9\}$

8.95

k	Observed N_k	Expected m_k	$(N_k - m_k)^2/m_k$
1	25	16	81/16
2	6	16	100/16
3	19	16	9/16
4	16	16	0/16
5	10	16	36/16
6	20	16	16/16

$$D^2 = \sum_k (N_k - m_k)^2/m_k = 242/16 = 15.125 > 11.07$$

$\Rightarrow$ Reject hypothesis.

8.96 Suppose N pairs of numbers are generated.

1. Partition the unit square into K disjoint subregions of equal area such that

$$\frac{N}{K} \geq 5 \Rightarrow N \geq 5K$$

2. Apply the Chi-Square test:

$$D^2 = \sum_{j=1}^{K} \frac{(N_j - N/K)^2}{N/K} \lesssim t_\alpha$$

where N_j is the number of pairs that fall in the jth region, and t_α is the threshold value determined by the significance level and the degrees of freedom $K - 1$.